Symmetry and Cauchy-completion

Extended abstract of a talk at the Séminaire Itinérant de Catégories
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1. Statement of the problem

A quantaloid $\mathcal{Q}$ is a category enriched in the symmetric monoidal closed category Sup of complete lattices and supremum-preserving functions. An *involution* on a quantaloid $\mathcal{Q}$ is a Sup-functor $(-)^{\mathsf{o}}\colon \mathcal{Q}^{\mathsf{op}} \longrightarrow \mathcal{Q}$ which is the identity on objects and satisfies $f^{\mathsf{oo}} = f$ for any morphism f in $\mathcal{Q}$. The pair $(\mathcal{Q}, (-)^{\mathsf{o}})$ is then said to form an *involutive quantaloid.* Whenever a morphism $f\colon A \longrightarrow B$ in a quantaloid (or in a locally ordered category, for that matter) is supposed to be a left adjoint, we write f^* for its right adjoint. In many examples there is a big difference between the involute f^{o} and the adjoint f^* of a given morphism f, so morphisms for which involute and adjoint coincide, deserve a name:

Definition 1.1 *In a quantaloid $\mathcal{Q}$ with involution $f \mapsto f^{\mathsf{o}}$, an* o-symmetric left adjoint *(or simply* symmetric left adjoint *if the context makes the involution clear) is a left adjoint whose right adjoint is its involute.*

Precisely as we write $\mathsf{Map}(\mathcal{Q})$ for the category of left adjoints in $\mathcal{Q}$ (this notation being motivated by the widespread use of the word "map" synonymously with "left adjoint"), we shall write $\mathsf{SymMap}(\mathcal{Q})$ for the category of symmetric left adjoints.

Viewing $\mathcal{Q}$ as a bicategory, it is natural to study categories, functors and distributors enriched in $\mathcal{Q}$. We write $\mathsf{Cat}(\mathcal{Q})$ for the 2-category of $\mathcal{Q}$-categories and $\mathcal{Q}$-functors, and $\mathsf{Dist}(\mathcal{Q})$ for the quantaloid of $\mathcal{Q}$-categories and $\mathcal{Q}$-distributors. Each functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ determines an adjoint pair of distributors: $\mathbb{B}(-, F-)\colon \mathbb{A} \multimap \mathbb{B}$, with elements $\mathbb{B}(y, Fx)$ for $(x, y) \in \mathbb{A}_0 \times \mathbb{B}_0$, is left adjoint to $\mathbb{B}(F-, -)\colon \mathbb{B} \multimap \mathbb{A}$ in the quantaloid $\mathsf{Dist}(\mathcal{Q})$. These distributors are said to be 'represented by F'. This amounts to a 2-functor

$$\mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Map}(\mathsf{Dist}(\mathcal{Q}))\colon \Big(F\colon \mathbb{A} \longrightarrow \mathbb{B}\Big) \mapsto \Big(\mathbb{B}(-, F-)\colon \mathbb{A} \multimap \mathbb{B}\Big). \tag{1}$$

A $\mathcal{Q}$-category $\mathbb{C}$ is said to be *Cauchy complete* [Lawvere, 1973] when for each $\mathcal{Q}$-category $\mathbb{X}$ the functor in (1) determines an equivalence $\mathsf{Cat}(\mathcal{Q})(\mathbb{X}, \mathbb{C}) \simeq \mathsf{Map}(\mathsf{Dist}(\mathcal{Q}))(\mathbb{X}, \mathbb{C})$. The full inclusion

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of the Cauchy complete $\mathcal{Q}$-categories into $\mathsf{Cat}(\mathcal{Q})$ admits a left adjoint:

$$\mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q}) \underset{\text{full incl.}}{\overset{(-)_{\mathsf{cc}}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathsf{Cat}(\mathcal{Q}). \tag{2}$$

Thus, each $\mathcal{Q}$-category $\mathbb{C}$ has a Cauchy completion $\mathbb{C}_{\mathsf{cc}}$, which can be computed explicitly as follows: objects are the left adjoint presheaves[1] on $\mathbb{C}$, the type of such a left adjoint $\phi\colon *_X \multimap \mathbb{C}$ is $X \in \mathcal{Q}$, and for another such $\psi\colon *_Y \multimap \mathbb{C}$ the hom-arrow $\mathbb{C}_{\mathsf{cc}}(\psi,\phi)\colon X \longrightarrow Y$ in $\mathcal{Q}$ is the single element of the composite distributor $\psi^* \otimes \phi$ (where $\psi \dashv \psi^*$).

If $\mathcal{Q}$ comes equipped with an involution, it makes sense to consider symmetric $\mathcal{Q}$-enriched categories:

Definition 1.2 (Betti and Walters, 1982) *Let $\mathcal{Q}$ be a small involutive quantaloid, with involution $f \mapsto f^{\circ}$. A $\mathcal{Q}$-category $\mathbb{A}$ is* symmetric *when $\mathbb{A}(x,y) = \mathbb{A}(y,x)^{\circ}$ for every two objects $x, y \in \mathbb{A}$.*

We shall write $\mathsf{SymCat}(\mathcal{Q})$ for the full sub-2-category of $\mathsf{Cat}(\mathcal{Q})$ determined by the symmetric $\mathcal{Q}$-categories; it is easy to see that the local order in $\mathsf{SymCat}(\mathcal{Q})$ is in fact symmetric (but not anti-symmetric). The full embedding $\mathsf{SymCat}(\mathcal{Q}) \hookrightarrow \mathsf{Cat}(\mathcal{Q})$ has a right adjoint:

$$\mathsf{SymCat}(\mathcal{Q}) \underset{(-)_{\mathsf{s}}}{\overset{\text{full incl.}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{Cat}(\mathcal{Q}). \tag{3}$$

This 'symmetrisation' sends a $\mathcal{Q}$-category $\mathbb{C}$ to the symmetric $\mathcal{Q}$-category $\mathbb{C}_{\mathsf{s}}$ whose objects (and types) are those of $\mathbb{C}$, but for any two objects x, y the hom-arrow is

$$\mathbb{C}_{\mathsf{s}}(y,x) := \mathbb{C}(y,x) \wedge \mathbb{C}(x,y)^{\circ}.$$

The counit of this adjunction has components $S_{\mathbb{C}}\colon \mathbb{C}_{\mathsf{s}} \longrightarrow \mathbb{C}\colon x \mapsto x$.

R. Betti and B. Walters [1982] raised the question "whether the Cauchy completion of a symmetric [quantaloid-enriched] category is again symmetric". That is to say, they ask whether it is possible to *restrict* the Cauchy completion functor $(-)_{\mathsf{cc}}\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ along the embedding $\mathsf{SymCat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ of symmetric $\mathcal{Q}$-categories:

$$\begin{array}{ccc}
\mathsf{Cat}(\mathcal{Q}) & \xrightarrow{(-)_{\mathsf{cc}}} & \mathsf{Cat}(\mathcal{Q}) \\
{\scriptstyle \text{full incl.}}\uparrow & & \uparrow{\scriptstyle \text{full incl.}} \\
\mathsf{SymCat}(\mathcal{Q}) & \xrightarrow[?]{\cdots\cdots} & \mathsf{SymCat}(\mathcal{Q})
\end{array}$$

[1] A 'presheaf' on $\mathbb{A}$ is a distributor into $\mathbb{A}$ whose domain is a one-object category with an identity hom-arrow. Writing $*_X$ for the one-object $\mathcal{Q}$-category whose single object $*$ has type $X \in \mathcal{Q}_0$ and whose single hom-arrow is the identity 1_X, a presheaf is then typically written as $\phi\colon *_X \multimap \mathbb{A}$. (These are really the *contravariant* presheaves on $\mathbb{A}$; the *covariant* presheaves are the distributors from $\mathbb{A}$ to $*_X$. However, we shall only consider contravariant presheaves.)

They show that the answer to their question is affirmative for any "small quantaloid of relations" $\mathcal{R}(\mathcal{C},J)$ [Walters, 1982] as well as for Lawvere's quantale of non-negative reals $[0,\infty]$ [Lawvere, 1973], by giving an *ad hoc* proof in each case; but they also give an example of an involutive quantale for which the answer to their question is negative. Thus, it depends on the base quantaloid $\mathcal{Q}$ whether or not the Cauchy completion of a symmetric $\mathcal{Q}$-category is again symmetric.

In what follows, we address this issue in a slightly different manner to produce a single, simple argument for both Walters' small quantaloids of relations and Lawvere's quantale of non-negative real numbers, thus giving perhaps a more decisive answer to Betti and Walters' question.

2. Statement of our solution

We shall write $\mathsf{SymDist}(\mathcal{Q})$ for the full subquantaloid of $\mathsf{Dist}(\mathcal{Q})$ determined by the symmetric $\mathcal{Q}$-categories. It is easily verified that the involution $f\mapsto f^{\circ}$ on the base quantaloid $\mathcal{Q}$ extends to the quantaloid $\mathsf{SymDist}(\mathcal{Q})$: explicitly, if $\Phi\colon\mathbb{A}\xrightarrow{\circ}\mathbb{B}$ is a distributor between symmetric $\mathcal{Q}$-categories, then so is $\Phi^{\circ}\colon\mathbb{B}\xrightarrow{\circ}\mathbb{A}$, with elements $\Phi^{\circ}(a,b):=\Phi(b,a)^{\circ}$. And if $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ is a functor between symmetric $\mathcal{Q}$-categories, then the left adjoint distributor represented by F has the particular feature that it is a symmetric left adjoint in $\mathsf{SymDist}(\mathcal{Q})$ (in the sense of Definition 1.1). That is to say, the functor in (1) restricts to the symmetric situation, giving a commutative diagram

$$\begin{array}{ccc}\mathsf{Cat}(\mathcal{Q}) & \longrightarrow & \mathsf{Map}(\mathsf{Dist}(\mathcal{Q}))\\ \text{incl.}\uparrow & & \uparrow\text{incl.}\\ \mathsf{SymCat}(\mathcal{Q}) & \longrightarrow & \mathsf{SymMap}(\mathsf{SymDist}(\mathcal{Q}))\end{array} \tag{4}$$

In analogy with the notion of Cauchy completeness of a $\mathcal{Q}$-category, which refers to the functor in the top row of the above diagram, we now define an appropriate notion of completeness for *symmetric* $\mathcal{Q}$-categories:

Definition 2.1 *Let $\mathcal{Q}$ be a small involutive quantaloid. A symmetric $\mathcal{Q}$-category $\mathbb{A}$ is* symmetrically complete *if, for any symmetric $\mathcal{Q}$-category $\mathbb{X}$, the functor in the bottom row of the diagram in (4) determines an equivalence* $\mathsf{SymCat}(\mathcal{Q})(\mathbb{X},\mathbb{A})\simeq\mathsf{SymMap}(\mathsf{SymDist}(\mathcal{Q}))(\mathbb{X},\mathbb{A})$.

The full inclusion of symetrically complete symmetric $\mathcal{Q}$-categories into $\mathsf{SymCat}(\mathcal{Q})$ admits a left adjoint:

$$\mathsf{SymCat}_{\mathsf{sc}}(\mathcal{Q}) \underset{\text{full incl.}}{\overset{(-)_{\mathsf{sc}}}{\rightleftarrows}} \mathsf{SymCat}(\mathcal{Q}). \quad (\perp) \tag{5}$$

Explicitly, for a symmetric $\mathcal{Q}$-category $\mathbb{A}$, its symmetric completion $\mathbb{A}_{\mathsf{sc}}$ is the full subcategory of $\mathbb{A}_{\mathsf{cc}}$ determined by the *symmetric* left adjoint presheaves.

It is clear from this construction that there is a natural transformation

$$\begin{array}{ccc} \mathsf{Cat}(\mathcal{Q}) & \xrightarrow{(-)_{\mathsf{cc}}} & \mathsf{Cat}(\mathcal{Q}) \\ {\scriptstyle \text{incl.}}\uparrow & \overset{K}{\Nwarrow} & \uparrow{\scriptstyle \text{incl.}} \\ \mathsf{SymCat}(\mathcal{Q}) & \xrightarrow[(-)_{\mathsf{sc}}]{} & \mathsf{SymCat}(\mathcal{Q}) \end{array} \tag{6}$$

whose components are the full embeddings $K_{\mathbb{A}}\colon \mathbb{A}_{\mathsf{sc}} \longrightarrow \mathbb{A}_{\mathsf{cc}}\colon \phi \longrightarrow \phi$ of which the very definition of the symmetric completion speaks. Computing its mate [Kelly and Street, 1974] we find a natural transformation

$$\begin{array}{ccc} \mathsf{Cat}(\mathcal{Q}) & \xrightarrow{(-)_{\mathsf{cc}}} & \mathsf{Cat}(\mathcal{Q}) \\ {\scriptstyle (-)_{\mathsf{s}}}\downarrow & \overset{L}{\Nearrow} & \downarrow{\scriptstyle (-)_{\mathsf{s}}} \\ \mathsf{SymCat}(\mathcal{Q}) & \xrightarrow[(-)_{\mathsf{sc}}]{} & \mathsf{SymCat}(\mathcal{Q}) \end{array} \tag{7}$$

whose component at $\mathbb{C}$ in $\mathsf{Cat}(\mathcal{Q})$ is $L_{\mathbb{C}}\colon (\mathbb{C}_{\mathsf{s}})_{\mathsf{sc}} \longrightarrow (\mathbb{C}_{\mathsf{cc}})_{\mathsf{s}}\colon \phi \mapsto \mathbb{C}(-, S_{\mathbb{C}}-) \otimes \phi$. (Recall that $S_{\mathbb{C}}\colon \mathbb{C}_{\mathsf{s}} \longrightarrow \mathbb{C}\colon x \mapsto x$ is the counit of the adjunction in the diagram (3).)

Our result, proved in detail in [Heymans and Stubbe, 2010], can now be summarized as:

Theorem 2.2 *For a small involutive quantaloid $\mathcal{Q}$, the following statements are equivalent:*

1. *the natural transformation L is an isomorphism:*

$$\begin{array}{ccc} \mathsf{Cat}(\mathcal{Q}) & \xrightarrow{(-)_{\mathsf{cc}}} & \mathsf{Cat}(\mathcal{Q}) \\ {\scriptstyle (-)_{\mathsf{s}}}\downarrow & \overset{L}{\Nearrow}\,\cong & \downarrow{\scriptstyle (-)_{\mathsf{s}}} \\ \mathsf{SymCat}(\mathcal{Q}) & \xrightarrow[(-)_{\mathsf{sc}}]{} & \mathsf{SymCat}(\mathcal{Q}) \end{array}$$

2. *there is a right adjoint to the inclusion* $\mathsf{SymMap}(\mathsf{SymDist}(\mathcal{Q})) \longrightarrow \mathsf{Map}(\mathsf{Dist}(\mathcal{Q}))$ *making the following two squares commute:*

$$\begin{array}{ccc} \mathsf{Cat}(\mathcal{Q}) & \longrightarrow & \mathsf{Map}(\mathsf{Dist}(\mathcal{Q})) \\ {\scriptstyle \text{incl.}}\uparrow \dashv \downarrow{\scriptstyle (-)_{\mathsf{s}}} & & {\scriptstyle \text{incl.}}\uparrow \dashv \downarrow \\ \mathsf{SymCat}(\mathcal{Q}) & \longrightarrow & \mathsf{SymMap}(\mathsf{SymDist}(\mathcal{Q})) \end{array}$$

3. *for each $(f_i\colon X \longrightarrow X_i, g_i\colon X_i \longrightarrow X)_{i \in I}$ in $\mathcal{Q}$,*

$$\left.\begin{array}{l} f_k \circ g_j \circ f_j \leq f_k \\ g_j \circ f_j \circ g_k \leq g_k \\ 1_X \leq \bigvee_i g_i \circ f_i \end{array}\right\} \Longrightarrow\ 1_X \leq \bigvee_i (g_i \wedge f_i^{\circ}) \circ (g_i^{\circ} \wedge f_i)$$

In fact, given any $\Phi\colon \mathbb{C} \mathrel{-\!\!\circ\!\!\to} \mathbb{D}$ in $\mathsf{Map}(\mathsf{Dist}(\mathcal{Q}))$, we can define $\Phi_{\mathsf{s}}\colon \mathbb{C}_{\mathsf{s}} \mathrel{-\!\!\circ\!\!\to} \mathbb{D}_{\mathsf{s}}$ in $\mathsf{SymDist}(\mathcal{Q})$ as

$$\Phi_{\mathsf{s}} := \Big(\mathbb{D}(S_{\mathbb{D}}-,-) \otimes \Phi \otimes \mathbb{C}(-,S_{\mathbb{C}}-)\Big) \wedge \Big(\mathbb{C}(S_{\mathbb{C}}-,-) \otimes \Phi^* \otimes \mathbb{D}(-,S_{\mathbb{D}}-)\Big)^{\circ}$$

The statements in the Theorem are all equivalent to:

4. If $\mathbb{C} \underset{\Phi^*}{\overset{\Phi}{\overset{\perp}{\rightleftarrows}}} \mathbb{D}$ in $\mathsf{Dist}(\mathcal{Q})$ then $\mathbb{C}_{\mathsf{s}} \underset{(\Phi_{\mathsf{s}})^{\circ}}{\overset{\Phi_{\mathsf{s}}}{\overset{\perp}{\rightleftarrows}}} \mathbb{D}_{\mathsf{s}}$ in $\mathsf{SymDist}(\mathcal{Q})$.

It is then precisely this mapping $\Phi \mapsto \Phi_{\mathsf{s}}$, turning any adjunction into a symmetric adjunction, that makes up the right adjoint of which the second statement in the above Theorem speaks.

A corollary of Theorem 2.2 contains an answer to R. Betti and B. Walters' [1982] question about the symmetry of the Cauchy completion of a symmetric category:

Corollary 2.3 *If $\mathcal{Q}$ is a small involutive quantaloid satisfying the equivalent conditions in Theorem 2.2, then the following diagrams commute:*

$$\begin{array}{ccc} \mathsf{Cat}(\mathcal{Q}) & \xrightarrow{(-)_{\mathsf{cc}}} & \mathsf{Cat}(\mathcal{Q}) \\ {\scriptstyle \text{incl.}}\uparrow & & \uparrow{\scriptstyle \text{incl.}} \\ \mathsf{SymCat}(\mathcal{Q}) & \xrightarrow[(-)_{\mathsf{cc}}]{} & \mathsf{SymCat}(\mathcal{Q}) \end{array} \qquad \begin{array}{ccc} \mathsf{Cat}(\mathcal{Q}) & \xrightarrow{(-)_{\mathsf{s}}} & \mathsf{Cat}(\mathcal{Q}) \\ {\scriptstyle \text{incl.}}\uparrow & & \uparrow{\scriptstyle \text{incl.}} \\ \mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q}) & \xrightarrow[(-)_{\mathsf{s}}]{} & \mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q}) \end{array}$$

This implies that, whenever $\mathcal{Q}$ satisfies the equivalent conditions in Theorem 2.2, there is a *distributive law* [Beck, 1969; Street, 1972; Power and Watanabe, 2002] of the Cauchy completion monad over the symmetrisation comonad on the category $\mathsf{Cat}(\mathcal{Q})$. It is a consequence of the general theory of distributive laws that the monad $(-)_{\mathsf{cc}}$ restricts to the category of $(-)_{\mathsf{s}}$-coalgebras, that the comonad $(-)_{\mathsf{s}}$ restricts to the category of $(-)_{\mathsf{cc}}$-algebras, and that the categories of (co)algebras for these restricted (co)monads are equivalent to each other and are further equivalent to the category of so-called λ-*bialgebras* [Power and Watanabe, 2002, Corollary 6.8]. In the case at hand, a λ-bialgebra is simply a $\mathcal{Q}$-category which is both symmetric and Cauchy-complete (the "λ-compatibility" between algebra and coalgebra structure is trivially satisfied), and a morphism between λ-bialgebras is simply a functor between such $\mathcal{Q}$-categories.

3. Some examples

As we shall point out below, many an interesting involutive quantaloid $\mathcal{Q}$ satisfies the following condition: for any family $(f_i\colon X \longrightarrow X_i, g_i\colon X_i \longrightarrow X)_{i\in I}$ of morphisms in $\mathcal{Q}$,

$$1_X \le \bigvee_i g_i \circ f_i \implies 1_X \le \bigvee_i (f_i^{\circ} \wedge g_i) \circ (f_i \wedge g_i^{\circ}). \tag{8}$$

Obviously, this condition implies (the third of) the equivalent conditions in Theorem 2.2.

Example 3.1 (Generalised metric spaces) The condition in (8) is satisfied by the integral and commutative quantale[2] $Q = ([0, \infty], \bigwedge, +, 0)$ with its trivial involution. This "explains" the well known fact that the Cauchy completion of a symmetric generalised metric space [Lawvere, 1973] is again symmetric.

Example 3.2 (Locales) Any locale $(L, \bigvee, \wedge, \top)$ is a commutative (hence trivially involutive) and integral quantale. It is easily checked that the condition in (8) holds for L. Splitting the idempotents of the Sup-monoid $(L, \wedge, \top)$ gives an integral quantaloid with an obvious involution, that also satisfies the condition in (8).

Example 3.3 (Groupoid-quantaloids) The free quantaloid $\mathcal{Q}(\mathcal{G})$ on a groupoid $\mathcal{G}$ comes with a *canonical involution* $S \mapsto S^{\circ} := \{s^{-1} \mid s \in S\}$. The condition in (8) holds for $\mathcal{Q}(\mathcal{G})$.

Example 3.4 (Commutative group-quantales with trivial involution) For a commutative group $(G, \cdot, 1)$, also the group-quantale $\mathcal{Q}(G)$ is commutative, and – in contrast with the above example – it can therefore be equipped with the *trivial involution* $S \mapsto S^{\circ} := S$. Betti and Walters [1982] gave a simple example of such a commutative group-quantale with trivial involution for which the Cauchy completion of a symmetric enriched category is not necessarily symmetric: Let $G = \{1, a, b\}$ be the commutative group defined by $a \cdot a = b$, $b \cdot b = a$ and $a \cdot b = 1$; then the pair $(\{a\}, \{b\})$ of elements of $\mathcal{Q}(G)$ does satisfy the premise but not the conclusion of the fourth of the four equivalent conditions in Theorem 2.2.

Example 3.5 (Quantaloids determined by small sites) If $(\mathcal{C}, J)$ is a small site, then we write $\mathcal{R}(\mathcal{C}, J)$ for the so-called small quantaloid of relations [Walters, 1982]: it always satisfies the condition in (8). Any locale L can be thought of as a site $(\mathcal{C}, J)$, where $\mathcal{C}$ is the ordered set L and J is its so-called canonical topology: $\mathcal{R}(\mathcal{C}, J)$ is then isomorphic (as involutive quantaloid) to the quantaloid obtained by splitting the idempotents in the Sup-monoid L. And if $\mathcal{G}$ is a small groupoid and J is the smallest Grothendieck topology on $\mathcal{G}$, then the quantaloid of relations $\mathcal{R}(\mathcal{G}, J)$ is isomorphic to the free quantaloid $\mathcal{Q}(\mathcal{G})$ with its canonical involution. Hence both Examples 3.2 and 3.3 are covered by the construction of the quantaloid $\mathcal{R}(\mathcal{C}, J)$ from a small site $(\mathcal{C}, J)$.

Example 3.6 (Locally localic and modular quantaloids) Following [Freyd and Scedrov, 1990] we say that a quantaloid $\mathcal{Q}$ is locally localic when each $\mathcal{Q}(X, Y)$ is a locale; and $\mathcal{Q}$ is modular if it is involutive and when for any morphisms $f\colon Z \longrightarrow Y$, $g\colon Y \longrightarrow X$ and $h\colon Z \longrightarrow X$ in $\mathcal{Q}$ we have $gf \wedge h \leq g(f \wedge g^{\circ}h)$ (or equivalently, $gf \wedge h \leq (g \wedge hf^{\circ})f$). (Here we write the composition in $\mathcal{Q}$ by juxtaposition to avoid overly bracketed expressions.) Every locally localic and modular quantaloid $\mathcal{Q}$ satisfies the condition in (8). Any small quantaloid of relations $\mathcal{R}(\mathcal{C}, J)$ is in fact locally localic and modular, and thus it satisfies the condition in (8), hence this example further generalises the previous one.

Example 3.7 (Sets and relations) The quantaloid Rel of sets and relations is not small, but it is involutive (the involute of a relation is its opposite: $R^{\circ} = \{(y, x) \mid (x, y) \in R\}$) and it does

[2] A quantale is, by definition, a one-object quantaloid. Obviously, a quantale Q is commutative if and only if the identity function $1_Q\colon Q \longrightarrow Q$ is an involution: it is the *trivial involution*.

satisfy the condition in (8) (and therefore also the third condition in Theorem 2.2). In fact, this holds for any quantaloid $\mathsf{Rel}(\mathcal{E})$ of internal relations in a Grothendieck topos $\mathcal{E}$, because it is modular and locally localic [Freyd and Scedrov, 1990].

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