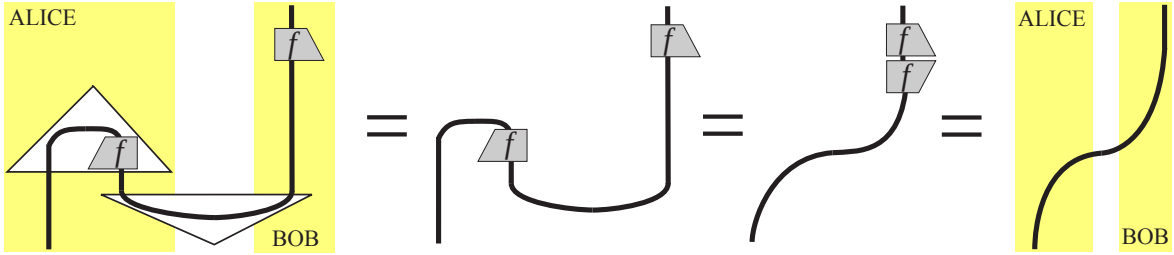
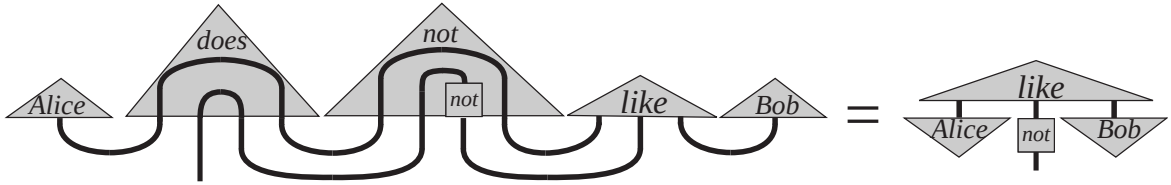
Selling Category Theory to the Masses

Bob Coecke – Quantum Group - Computer Science - Oxford University



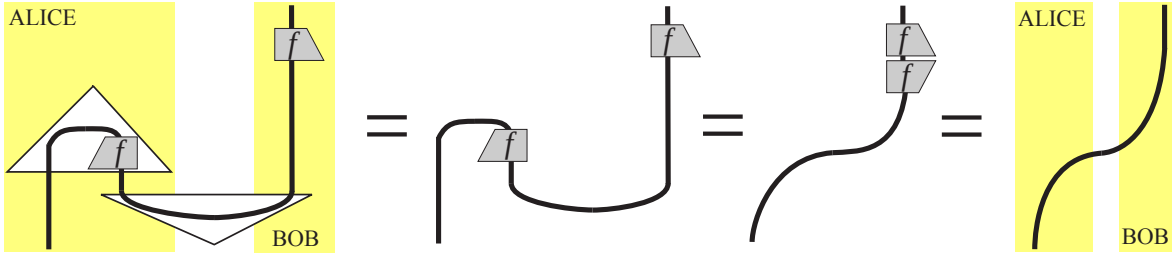


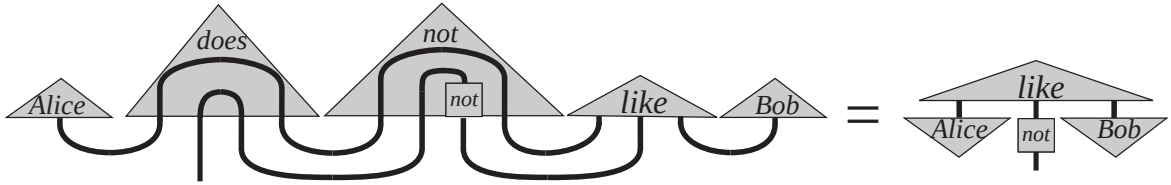
Samson Abramsky & BC (2004) *A categorical semantics for quantum protocols*. LiCS'04. arXiv:quant-ph/0402130 BC & **Eric O. Paquette** (2011) *Categories for the practicing physicist*. In: New Structures for Physics. arXiv:0905.3010 BC (2010) *Quantum picturalism*. Contemporary Physics **51**, 59–83. arXiv:0908.1787 BC & **Ross Duncan** (2011) *Interacting quantum observables*. New Journal of Physics **13**, 043016. arXiv:0906.4725 BC, **Mehrnoosh Sadrzadeh** & **Stephen Clark** (2011) *Mathematical foundations for a compositional distributional model of meaning*. Linguistic Analysis - Lambek Festschrift. arXiv:1003.4394 **Lucas Dixon**, R. Duncan, **Aleks Kissinger**, **Alex Merry** & **Matvey Soloviev**. `sites.google.com/site/quantomatic/`

EPSRC Engineering and Physical Sciences Research Council
Department of the Navy ONR Science & Technology
QICS
Information Society Technologies
Sixth Framework Programme
fq(x) Foundational Questions Institute
John Templeton Foundation
The Leverhulme Trust
AXA Protection Financière

. . . a tale of food, spiders and Google

Bob Coecke – Quantum Group - Computer Science - Oxford University





Samson Abramsky & BC (2004) *A categorical semantics for quantum protocols*. LiCS'04. arXiv:quant-ph/0402130 BC & **Eric O. Paquette** (2011) *Categories for the practicing physicist*. In: New Structures for Physics. arXiv:0905.3010 BC (2010) *Quantum picturalism*. Contemporary Physics **51**, 59–83. arXiv:0908.1787 BC & **Ross Duncan** (2011) *Interacting quantum observables*. New Journal of Physics **13**, 043016. arXiv:0906.4725 BC, **Mehrnoosh Sadrzadeh** & **Stephen Clark** (2011) *Mathematical foundations for a compositional distributional model of meaning*. Linguistic Analysis - Lambek Festschrift. arXiv:1003.4394 **Lucas Dixon**, R. Duncan, **Aleks Kissinger**, **Alex Merry** & **Matvey Soloviev**. `sites.google.com/site/quantomatic/`

EPSRC Engineering and Physical Sciences Research Council
Department of the Navy Office of Naval Research Science & Technology
QICS
Information Society Technologies
Sixth Framework Programme
fq(x) Foundational Questions Institute
John Templeton Foundation
The Leverhulme Trust
AXA Protection Financière

Task: selling Category Theory to the masses!

Task: selling Category Theory to the masses!

. . . is it just tedious abstract nonsense?

Task: selling Category Theory to the masses!

. . . is it just tedious abstract nonsense?

No! Categories are everywhere!

1. Let A be a raw potato.

1. Let A be a raw potato.

A admits many *states* e.g. dirty, clean, skinned, ...

1. Let A be a raw potato.

A admits many *states* e.g. dirty, clean, skinned, ...

2. We want to *process* A into cooked potato B.

B admits many *states* e.g. boiled, fried, deep fried, baked with skin, baked without skin, ...

1. Let A be a raw potato.

A admits many *states* e.g. dirty, clean, skinned, ...

2. We want to *process* A into cooked potato B.

B admits many *states* e.g. boiled, fried, deep fried, baked with skin, baked without skin, ... Let

$$A \xrightarrow{f} B \qquad A \xrightarrow{f'} B \qquad A \xrightarrow{f''} B$$

be boiling, frying, baking.

1. Let A be a raw potato.

A admits many *states* e.g. dirty, clean, skinned, ...

2. We want to *process* A into cooked potato B.

B admits many *states* e.g. boiled, fried, deep fried, baked with skin, baked without skin, ... Let

$$A \xrightarrow{f} B \qquad A \xrightarrow{f'} B \qquad A \xrightarrow{f''} B$$

be boiling, frying, baking. *States* are *processes*

$$\mathrm{I} := \text{unspecified} \xrightarrow{\psi} A.$$

3. Let

$$A \xrightarrow{g \circ f} C$$

be the *composite process* of first boiling $A \xrightarrow{f} B$ and then salting $B \xrightarrow{g} C$.

3. Let

$$A \xrightarrow{g \circ f} C$$

be the *composite process* of first boiling $A \xrightarrow{f} B$ and then salting $B \xrightarrow{g} C$. Let

$$X \xrightarrow{\mathbf{1}_X} X$$

be doing nothing. We have $\mathbf{1}_Y \circ \xi = \xi \circ \mathbf{1}_X = \xi$.

4. Let $A \otimes D$ be potato A and carrot D

4. Let $A \otimes D$ be potato A and carrot D and let

$$A \otimes D \xrightarrow{f \otimes h} B \otimes E$$

be boiling potato while frying carrot.

4. Let $A \otimes D$ be potato A and carrot D and let

$$A \otimes D \xrightarrow{f \otimes h} B \otimes E$$

be boiling potato while frying carrot. Let

$$C \otimes F \xrightarrow{x} M$$

be mashing spice-cook-potato and spice-cook-carrot.

5. Total *process*:

$$A \otimes D \xrightarrow{f \otimes h} B \otimes E \xrightarrow{g \otimes k} C \otimes F \xrightarrow{x} M = A \otimes D \xrightarrow{x \circ (g \otimes k) \circ (f \otimes h)} M.$$

5. Total *process*:

$$A \otimes D \xrightarrow{f \otimes h} B \otimes E \xrightarrow{g \otimes k} C \otimes F \xrightarrow{x} M = A \otimes D \xrightarrow{x \circ (g \otimes k) \circ (f \otimes h)} M.$$

6. *Recipe* = *composition structure* on *processes*.

5. Total *process*:

$$A \otimes D \xrightarrow{f \otimes h} B \otimes E \xrightarrow{g \otimes k} C \otimes F \xrightarrow{x} M = A \otimes D \xrightarrow{x \circ (g \otimes k) \circ (f \otimes h)} M.$$

6. *Recipe* = *composition structure* on *processes*.

7. *Laws governing recipes*:

$$(\mathbf{1}_B \otimes g) \circ (f \otimes \mathbf{1}_C) = (f \otimes \mathbf{1}_D) \circ (\mathbf{1}_A \otimes g)$$

5. Total *process*:

$$A \otimes D \xrightarrow{f \otimes h} B \otimes E \xrightarrow{g \otimes k} C \otimes F \xrightarrow{x} M = A \otimes D \xrightarrow{x \circ (g \otimes k) \circ (f \otimes h)} M.$$

6. *Recipe* = *composition structure* on *processes*.

7. *Laws governing recipes*:

$$(\mathbf{1}_B \otimes g) \circ (f \otimes \mathbf{1}_C) = (f \otimes \mathbf{1}_D) \circ (\mathbf{1}_A \otimes g)$$

i.e.

boil potato then fry carrot = fry carrot then boil potato

5. Total *process*:

$$A \otimes D \xrightarrow{f \otimes h} B \otimes E \xrightarrow{g \otimes k} C \otimes F \xrightarrow{x} M = A \otimes D \xrightarrow{x \circ (g \otimes k) \circ (f \otimes h)} M.$$

6. *Recipe* = *composition structure* on *processes*.

7. *Laws governing recipes*:

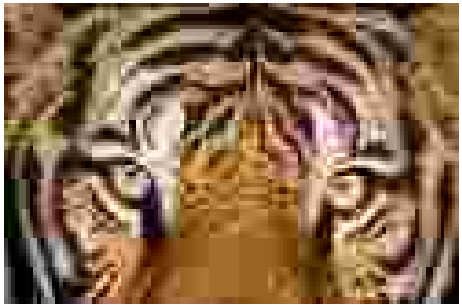
$$(\mathbf{1}_B \otimes g) \circ (f \otimes \mathbf{1}_C) = (f \otimes \mathbf{1}_D) \circ (\mathbf{1}_A \otimes g)$$

i.e.

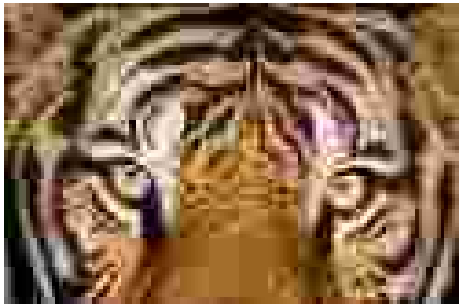
boil potato then fry carrot = fry carrot then boil potato

$\Rightarrow$ **Symmetric Monoidal Category**

— *Why does a tiger have stripes and a lion doesn't?* —



— *Why does a tiger have stripes and a lion doesn't?* —



prey $\otimes$ predator $\otimes$ environment

$\downarrow$ hunt

dead prey $\otimes$ eating predator

BOXES AND WIRES

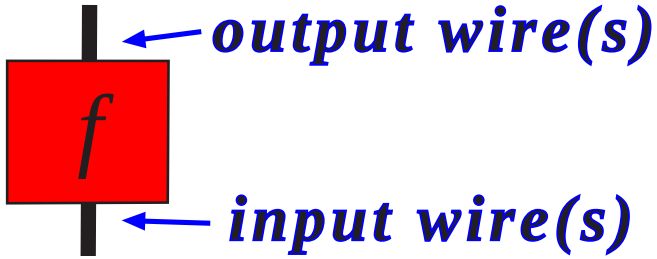
Roger Penrose (1971) *Applications of negative dimensional tensors*. In: Combinatorial Mathematics and its Applications, 221–244. Academic Press.

André Joyal and Ross Street (1991) *The Geometry of tensor calculus* I. Advances in Mathematics **88**, 55–112.

Bob Coecke and Eric Oliver Paquette (2011) *Categories for the practicing physicist*. In: New Structures for Physics, B. Coecke (ed), Springer-Verlag. `arXiv:0905.3010`

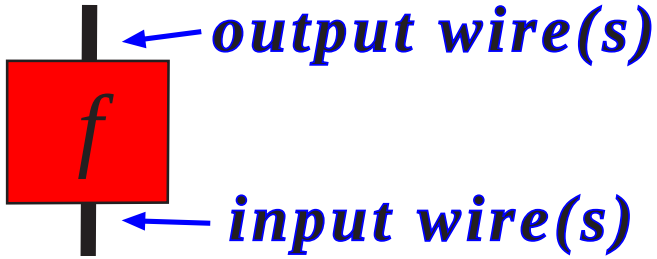
— *wire and box language* —

Box :=



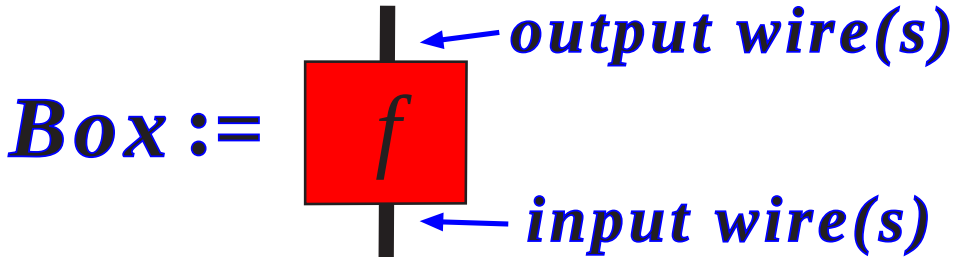
— *wire and box language* —

Box :=



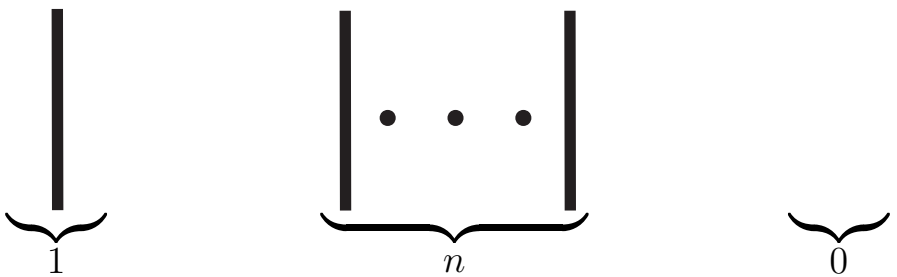
Interpretation: wire := system ; box := **process**

— *wire and box language* —



Interpretation: wire := system ; box := **process**

one system: n *sub*systems: no system:

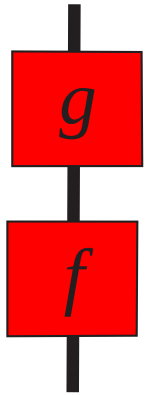


— *wire and box games* —

— *wire and box games* —

sequential or *causal* or *connected* composition:

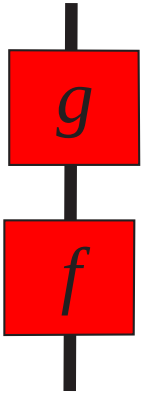
$g \circ f \equiv$



— *wire and box games* —

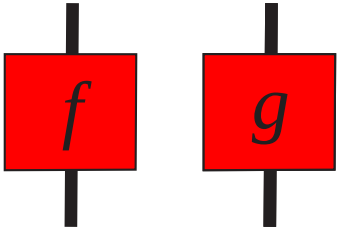
sequential or *causal* or *connected* composition:

$g \circ f \equiv$



parallel or *acausal* or *disconnected* composition:

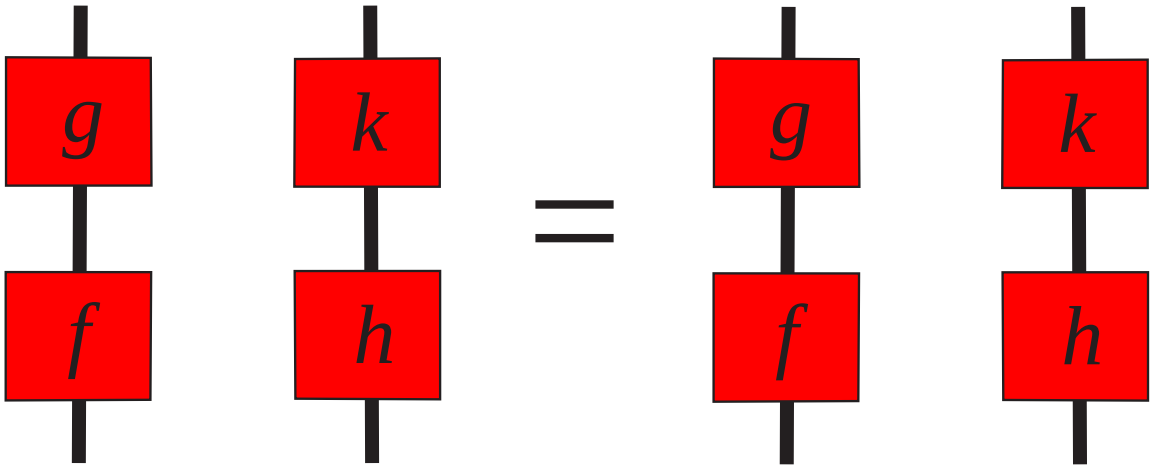
$f \otimes g \equiv$



— *merely a new notation?* —

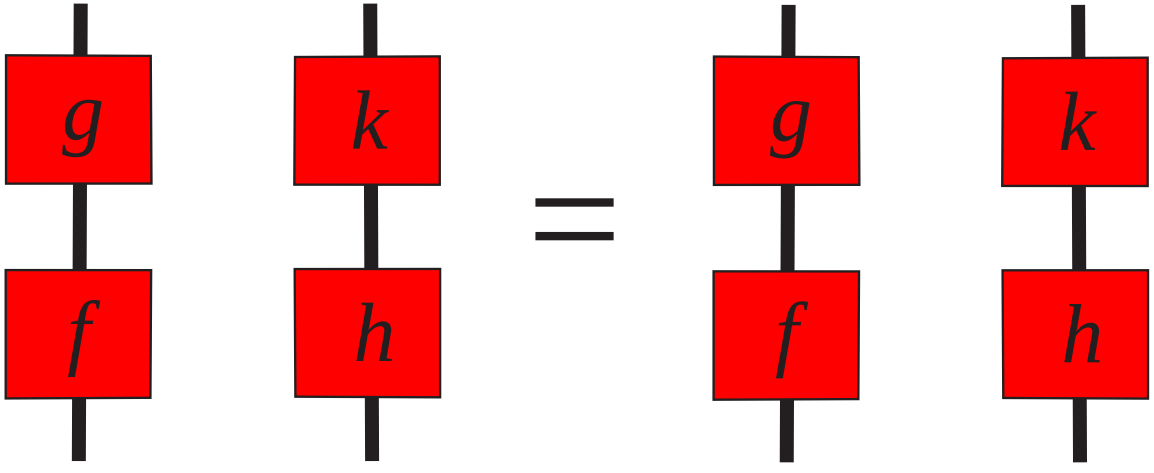
— *merely a new notation?* —

$$(g \circ f) \otimes (k \circ h) = (g \otimes k) \circ (f \otimes h)$$



— *merely a new notation?* —

$$(g \circ f) \otimes (k \circ h) = (g \otimes k) \circ (f \otimes h)$$

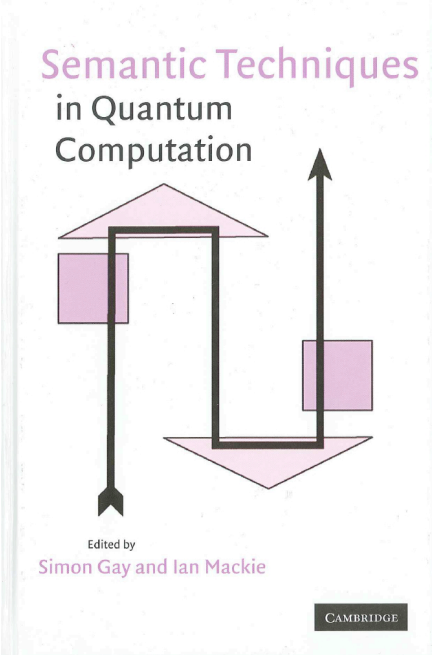


peel potato and then fry it,
while,
clean carrot and then boil it

=

peel potato while clean carrot,
and then,
fry potato while boil carrot

MINIMAL QUANTUM PROCESS LANGUAGE



— *genesis* —

[von Neumann 1932] Formalized quantum mechanics in "Mathematische Grundlagen der Quantenmechanik"

— *genesis* —

[von Neumann 1932] Formalized quantum mechanics in "Mathematische Grundlagen der Quantenmechanik"

[von Neumann to Birkhoff 1935] "I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space no more." (sic)

— *genesis* —

[von Neumann 1932] Formalized quantum mechanics in "Mathematische Grundlagen der Quantenmechanik"

[von Neumann to Birkhoff 1935] "I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space no more." (sic)

[Birkhoff and von Neumann 1936] **The Logic of Quantum Mechanics** in *Annals of Mathematics*.

— *genesis* —

[von Neumann 1932] Formalized quantum mechanics in "Mathematische Grundlagen der Quantenmechanik"

[von Neumann to Birkhoff 1935] "I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space no more." (sic)

[Birkhoff and von Neumann 1936] **The Logic of Quantum Mechanics** in *Annals of Mathematics*.

[1936 – 2000] many followed them, ...

— *genesis* —

[von Neumann 1932] Formalized quantum mechanics in "Mathematische Grundlagen der Quantenmechanik"

[von Neumann to Birkhoff 1935] "I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space no more." (sic)

[Birkhoff and von Neumann 1936] **The Logic of Quantum Mechanics** in *Annals of Mathematics*.

[1936 – 2000] many followed them, ... and **FAILED**.

— *the mathematics of it* —

— *the mathematics of it* —

Hilber space stuff: continuum, field structure of complex numbers, vector space over it, inner-product, etc.

— *the mathematics of it* —

Hilber space stuff: continuum, field structure of complex numbers, vector space over it, inner-product, etc.

WHY?

— *the mathematics of it* —

Hilber space stuff: continuum, field structure of complex numbers, vector space over it, inner-product, etc.

WHY?

von Neumann: only used *it* since *it* was ‘available’.

— *the physics of it* —

— *the physics of it* —

von Neumann crafted Birkhoff-von Neumann Quantum 'Logic' to capture the concept of **superposition**.

— *the physics of it* —

von Neumann crafted Birkhoff-von Neumann Quantum 'Logic' to capture the concept of **superposition**.

Schrödinger (1935): the stuff which is the true soul of quantum theory is **how quantum systems compose**.

— *the physics of it* —

von Neumann crafted Birkhoff-von Neumann Quantum 'Logic' to capture the concept of **superposition**.

Schrödinger (1935): the stuff which is the true soul of quantum theory is **how quantum systems compose**.

Quantum Computer Scientists: Schrödinger is right!

— *the game plan* —

— *the game plan* —

Task 0. Solve:

$$\frac{\texttt{tensor product structure}}{\texttt{the other stuff}} = ???$$

— *the game plan* —

Task 0. Solve:

$$\frac{\texttt{tensor product structure}}{\texttt{the other stuff}} = ???$$

i.e. **axiomatize** "$\otimes$" **without reference to spaces**.

— *the game plan* —

Task 0. Solve:

$$\frac{\texttt{tensor product structure}}{\texttt{the other stuff}} = ???$$

i.e. **axiomatize** "$\otimes$" **without reference to spaces**.

Task 1. Investigate which assumptions (i.e. which structure) on $\otimes$ is needed to deduce **physical phenomena**.

— *the game plan* —

Task 0. Solve:

$$\frac{\texttt{tensor product structure}}{\texttt{the other stuff}} = ???$$

i.e. **axiomatize** "$\otimes$" **without reference to spaces**.

Task 1. Investigate which assumptions (i.e. which structure) on $\otimes$ is needed to deduce **physical phenomena**.

Task 2. Investigate wether such an "interaction structure" appear elsewhere in **"our classical reality"**.

Outcome 1a:

Outcome 1a: "Sheer ratio of results to assumptions"

Hans Halvorson (2010) Editorial to: *Deep Beauty: Understanding the Quantum World through Mathematical Innovation*, Cambridge University Press.

Outcome 1a: "Sheer ratio of results to assumptions" confirms that we are probing something very essential.

Hans Halvorson (2010) Editorial to: *Deep Beauty: Understanding the Quantum World through Mathematical Innovation*, Cambridge University Press.

Outcome 1a: **"Sheer ratio of results to assumptions"** confirms that we are probing something very essential.

Outcome 1b: Exposing this structure has already helped to **solve open problems elsewhere**. (e.g. $2\times$ ICALP'10)

E.g.: Ross Duncan & Simon Perdrix (2010) *Rewriting measurement-based quantum computations with generalised flow*. ICALP'10.

Outcome 1a: **"Sheer ratio of results to assumptions"** confirms that we are probing something very essential.

Outcome 1b: Exposing this structure has already helped to **solve open problems elsewhere**. (e.g. $2\times$ ICALP'10)

Outcome 1c: **Framework is a simple intuitive (but rigorous) diagrammatic language**,

[1] BC (2010) ***Quantum picturalism***. Contemporary Physics **51**, 59–83.

Outcome 1a: **"Sheer ratio of results to assumptions"** confirms that we are probing something very essential.

Outcome 1b: Exposing this structure has already helped to **solve open problems elsewhere**. (e.g. $2\times$ ICALP'10)

Outcome 1c: **Framework is a simple intuitive (but rigorous) diagrammatic language**, meanwhile adopted by others e.g. Lucien Hardy in arXiv:1005.5164:

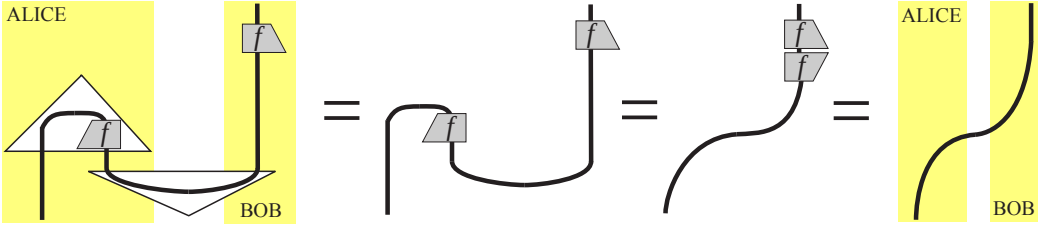
"... we join the ***quantum picturalism*** revolution [1]"

---

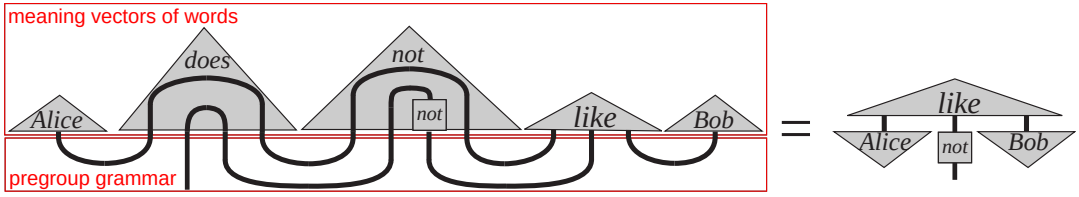
[1] BC (2010) ***Quantum picturalism***. Contemporary Physics **51**, 59–83.

Outcome 2a:

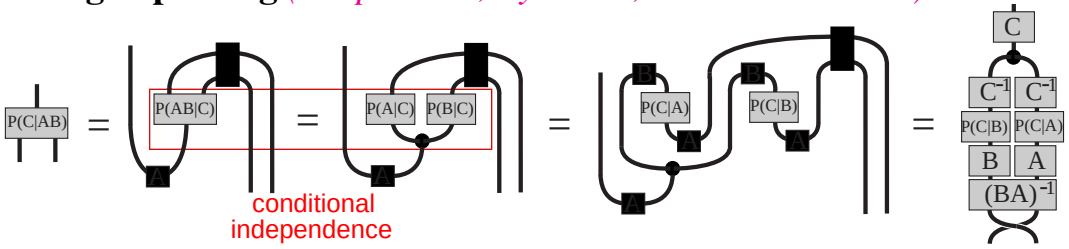
Behaviors of matter *(Abramsky-C; LiCS'04, quant-ph/0402130)* :



Meaning in language *(Clark-C-Sadrzadeh; Linguistic Analysis, arXiv:1003.4394)* :



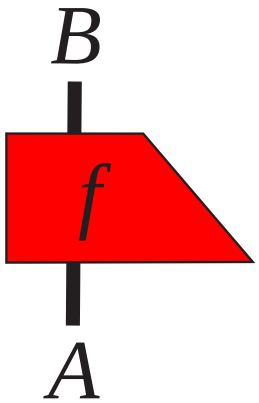
Knowledge updating *(C-Spekkens; Synthese, arXiv:1102.2368)* :



BOXES AND WIRES II

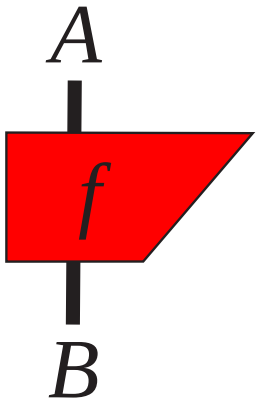
— *quantitative metric* —

$$f : A \to B$$



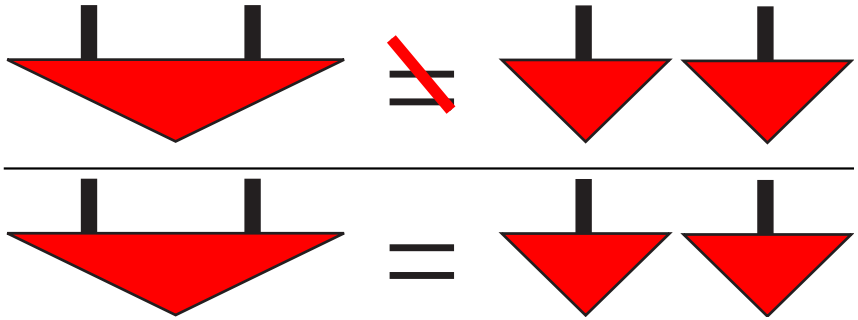
— *quantitative metric* —

$$f^{\dagger} : B \to A$$



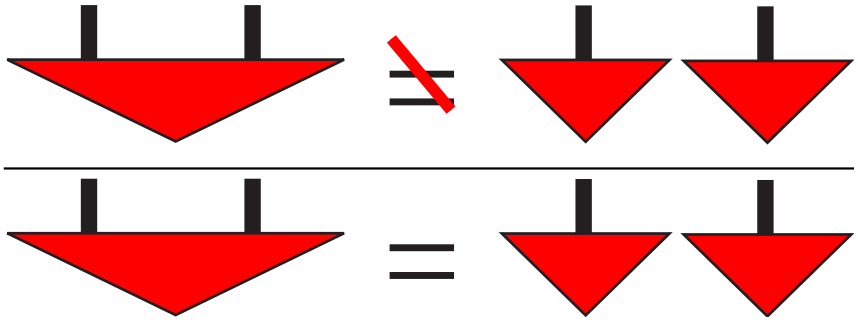
— *asserting (pure) entanglement* —

$$\frac{quantum}{classical} =$$



— *asserting (pure) entanglement* —

$$\frac{quantum}{classical} =$$



$\Rightarrow$ **introduce 'parallel wire' between systems:**

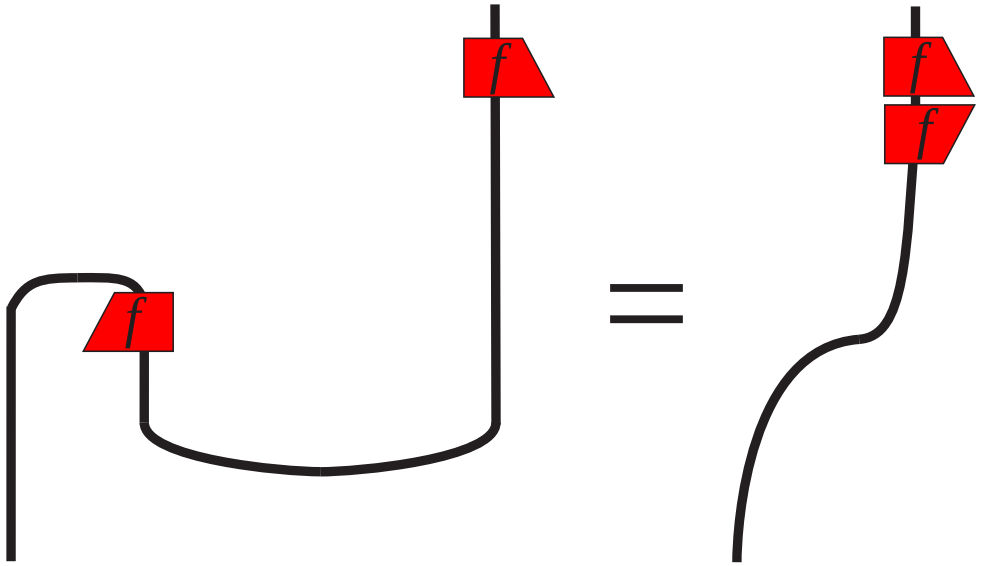
subject to: only topology matters!

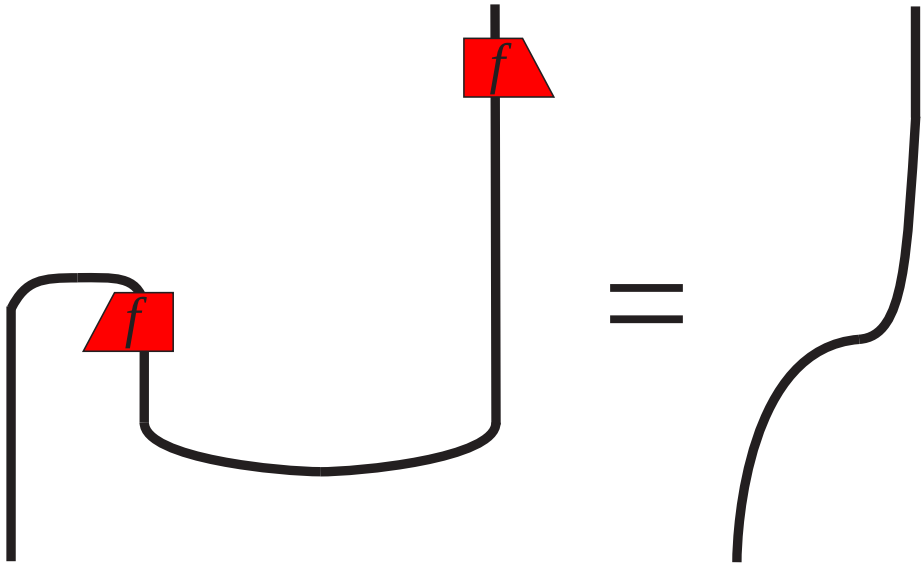
— *quantum-like* —

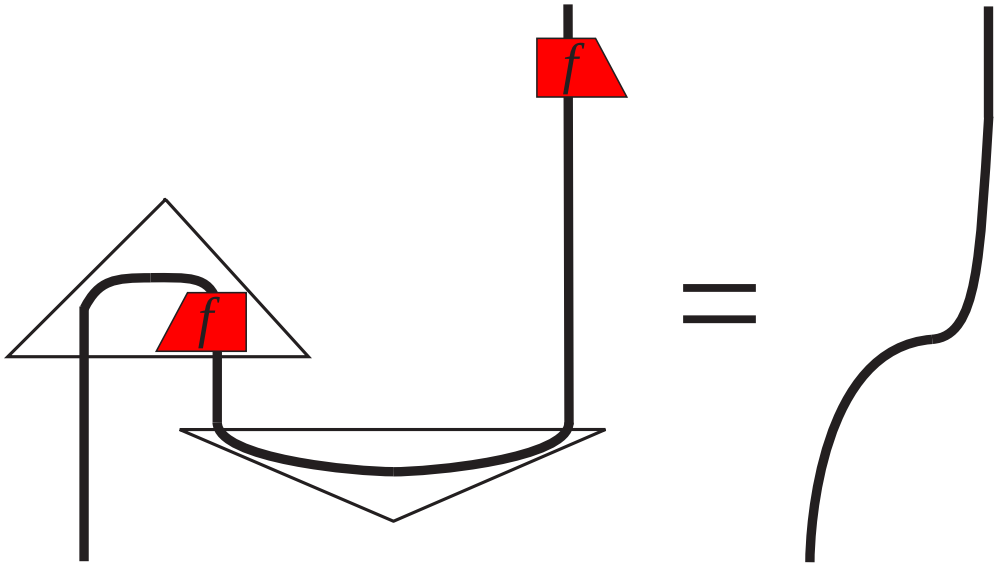
E.g.

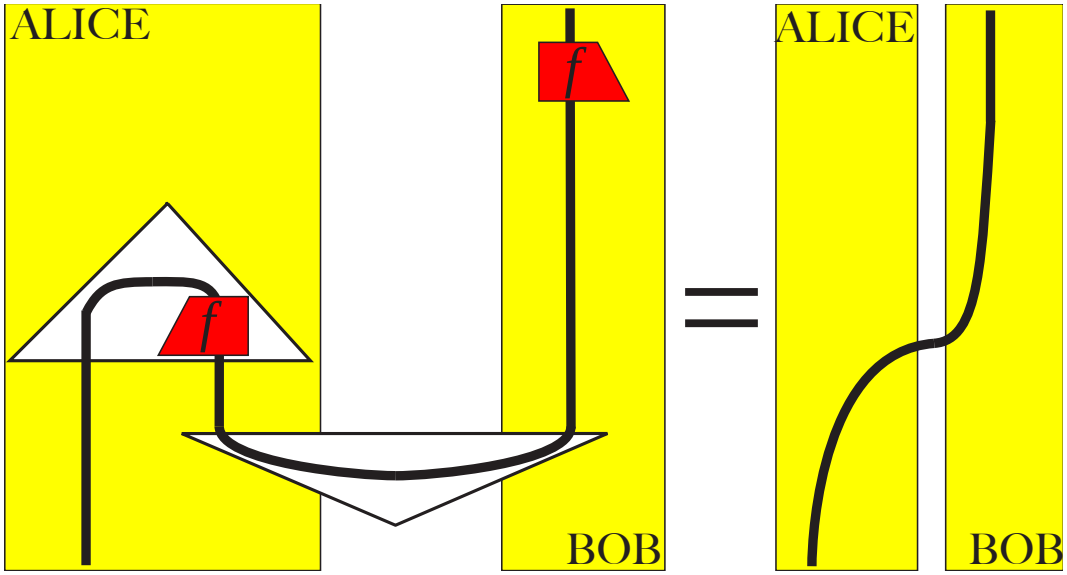
Transpose:

Conjugate:


f
f
=
f
f


f
f
=


f
f



⇒ quantum teleportation

— *symbolically: dagger compact categories* —

Thm. [Kelly-Laplaza '80; Selinger '05] *An equational statement between expressions in dagger compact categorical language holds if and only if it is derivable in the graphical notation via homotopy*.

Thm. [Hasegawa-Hofmann-Plotkin; Selinger '08] *An equational statement between expressions in dagger compact categorical language holds if and only if it is derivable in the dagger compact category of finite dimensional Hilbert spaces, linear maps, tensor product and adjoints*.

— *kindergarten quantum mechanics: the experiment* —

Contest in problem solving between:

- **Children using quantum picturalism**
- **Physics teachers using ordinary QM**

I expect the children to win!

[1] BC (2010) *Quantum picturalism*. Contemporary Physics 51, 59–83.

A SLIGHTLY DIFFERENT LANGUAGE FOR NATURAL LANGUAGE MEANING

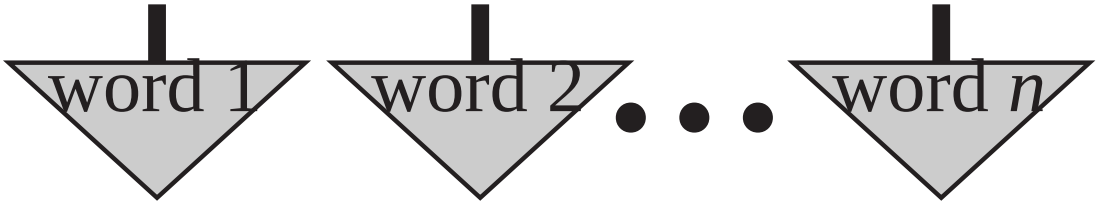
BC, Mehrnoosh Sadrzadeh & Stephen Clark (2010) *Mathematical foundations for a compositional distributional model of meaning*. `arXiv:1003.4394`

BC (2012) *The logic of quantum mechanics – Take II*. `arXiv:1204.3458`

— *the from-words-to-a-sentence process* —

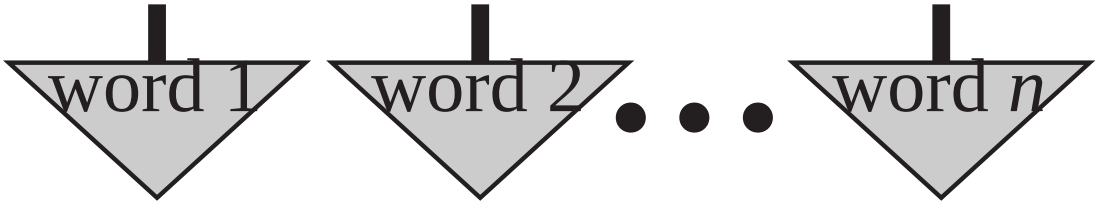
— *the from-words-to-a-sentence process* —

Consider meanings of **words**, e.g. as vectors (cf. Google):



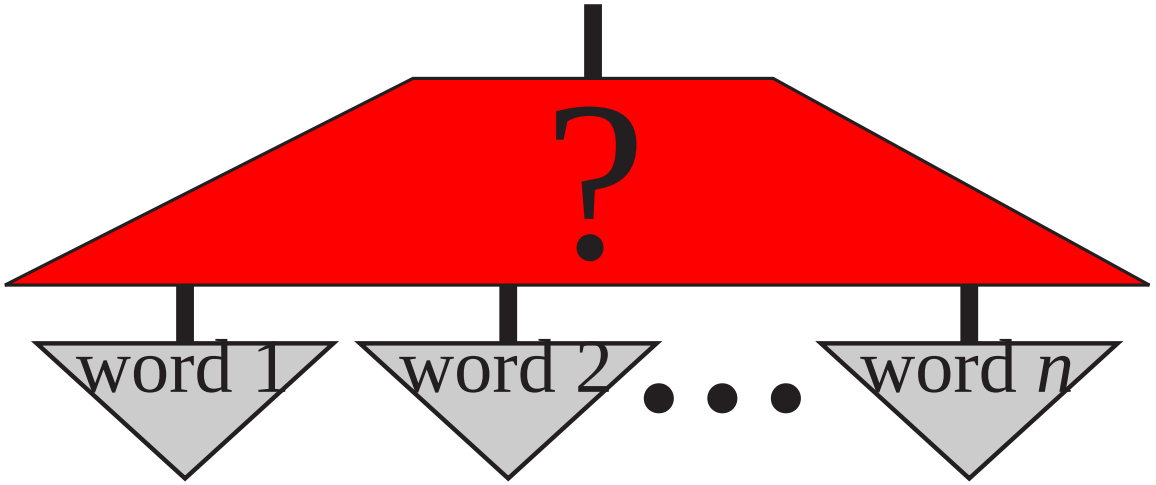
— *the from-words-to-a-sentence process* —

What is the meaning the **sentence** made up of these?



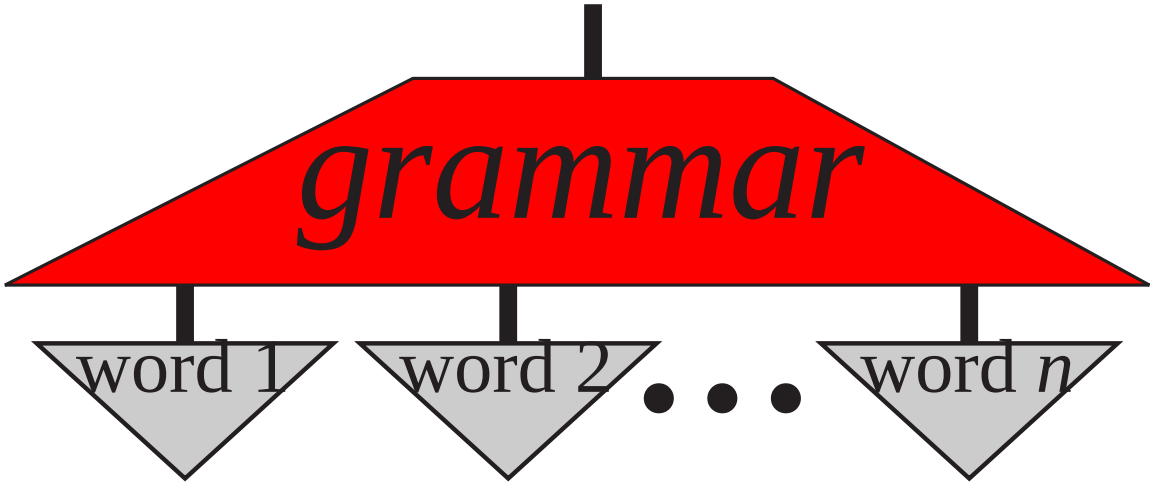
— *the from-words-to-a-sentence process* —

I.e. how do we/machines produce meanings of **sentences**?



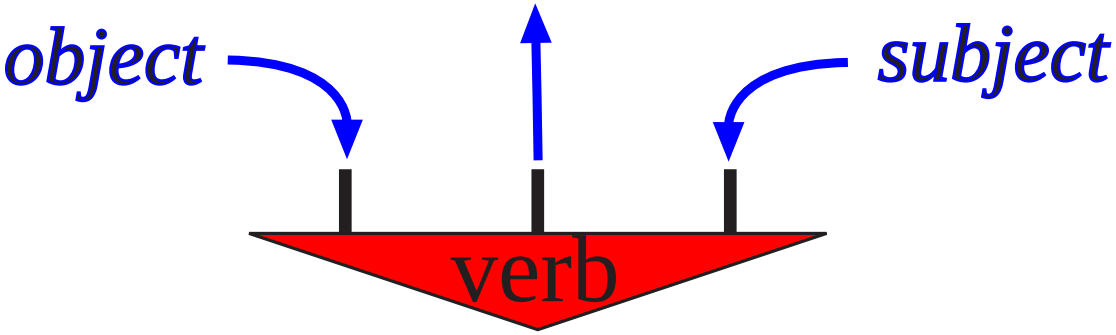
— *the from-words-to-a-sentence process* —

I.e. how do we/machines produce meanings of **sentences**?



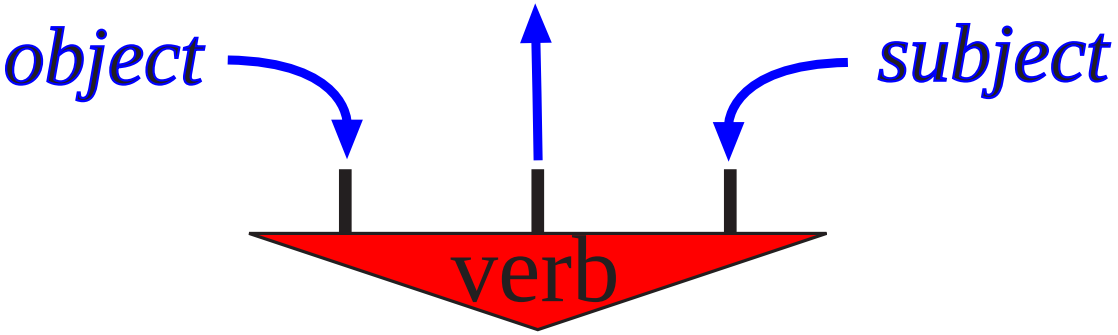
Gerald Gazdar (1996) *Paradigm merger in natural language processing*. In: Computing tomorrow: future research directions in computer science, eds., I. Wand and R. Milner, Cambridge University Press.

Information flow within a verb:

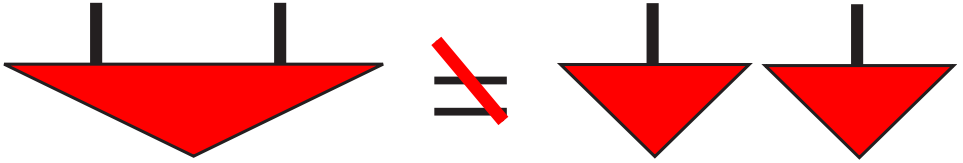


— *the from-words-to-a-sentence process* —

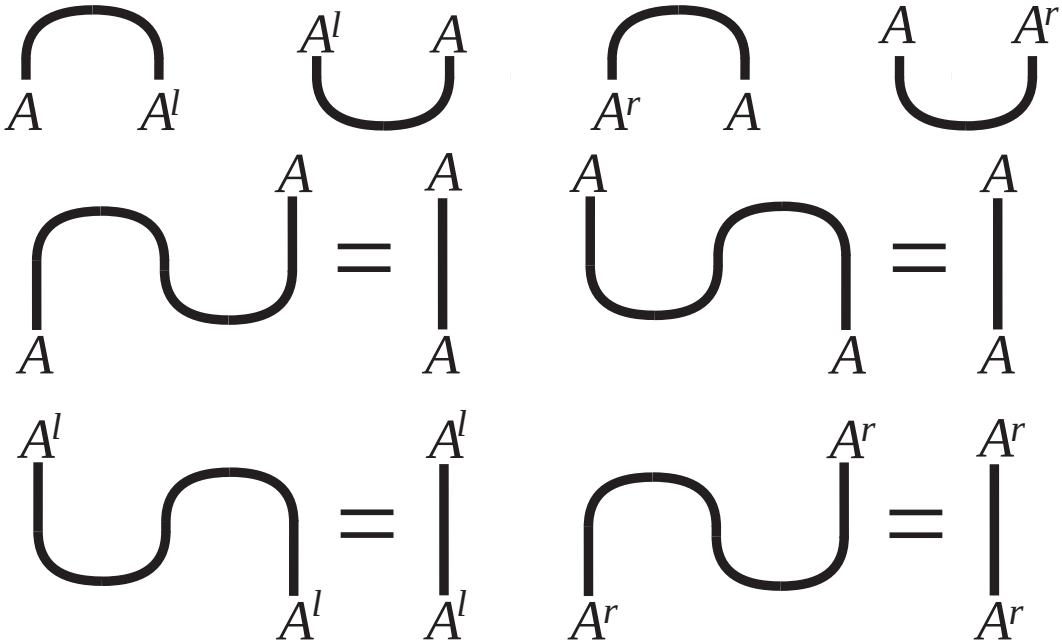
Information flow within a verb:



Again we have:



— *grammar as pregroups – Lambek '99* —



— *grammar as pregroups – Lambek '99* —

For noun type n, verb type is ${}^{-1}(n) \cdot s \cdot (n)^{-1}$, so:

— *grammar as pregroups – Lambek '99* —

For noun type n, verb type is ${}^{-1}(n) \cdot s \cdot (n)^{-1}$, so:

$$n \cdot {}^{-1}(n) \cdot s \cdot (n)^{-1} \cdot n = s$$

— *grammar as pregroups – Lambek '99* —

For noun type n, verb type is ${}^{-1}(n) \cdot s \cdot (n)^{-1}$, so:

$$n \cdot {}^{-1}(n) \cdot s \cdot (n)^{-1} \cdot n = s$$

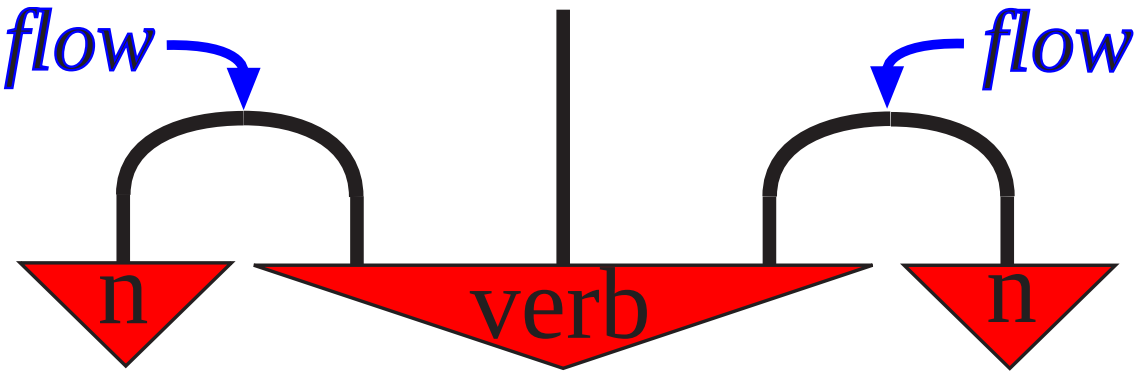
Diagrammatic typing:

— *grammar as pregroups – Lambek '99* —

For noun type n, verb type is ${}^{-1}(n) \cdot s \cdot (n)^{-1}$, so:

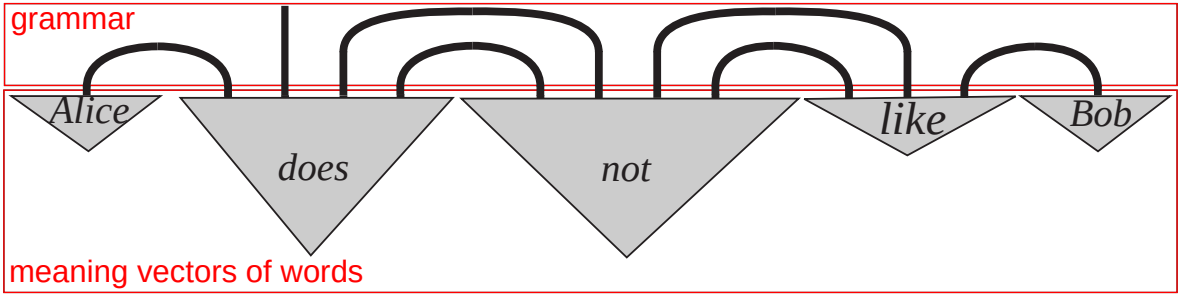
$$n \cdot {}^{-1}(n) \cdot s \cdot (n)^{-1} \cdot n = s$$

Diagrammatic meaning:

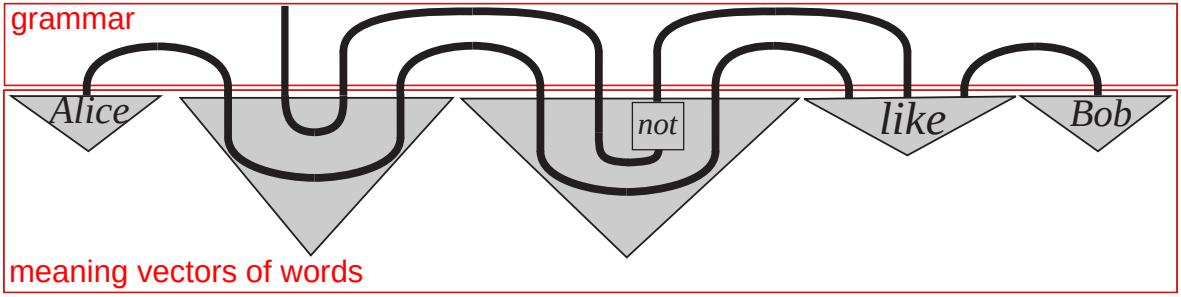


$$- \overrightarrow{Alice} \otimes \overrightarrow{does} \otimes \overrightarrow{not} \otimes \overrightarrow{like} \otimes \overrightarrow{Bob} -$$

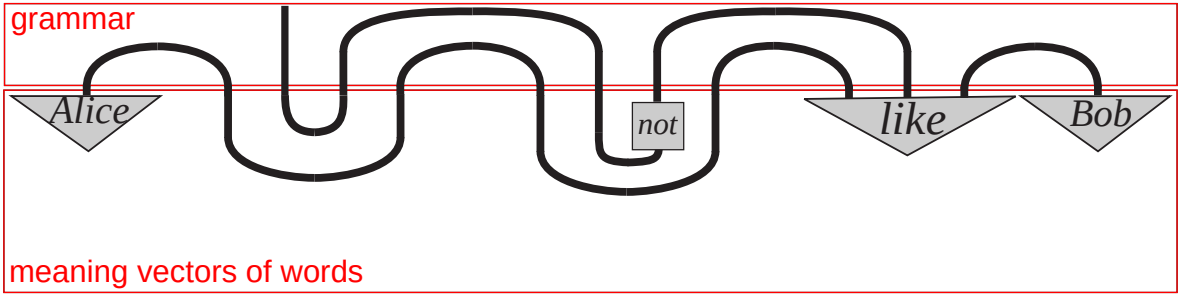
$$— \overrightarrow{Alice} \otimes \overrightarrow{does} \otimes \overrightarrow{not} \otimes \overrightarrow{like} \otimes \overrightarrow{Bob} —$$



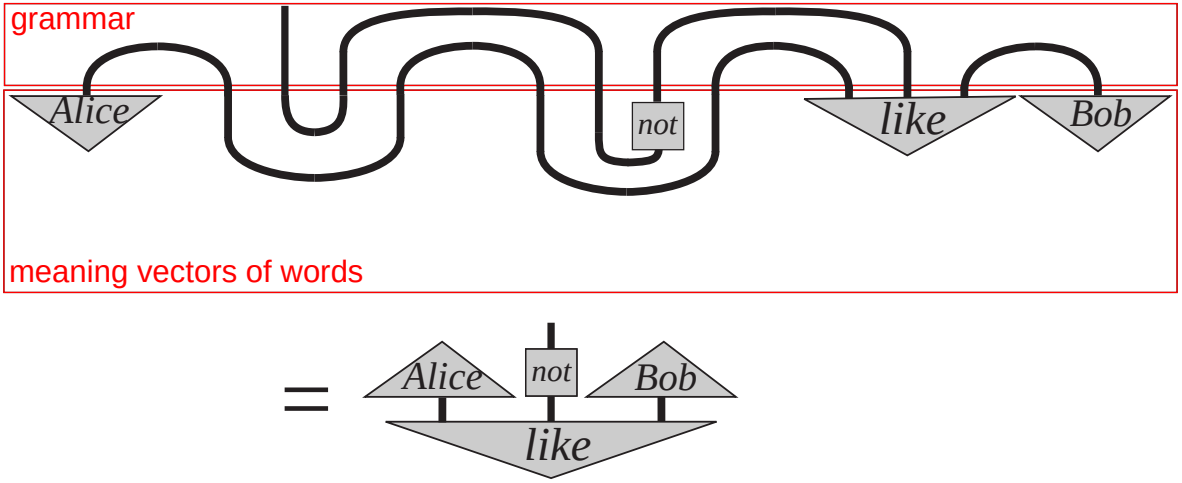
$$— \overrightarrow{Alice} \otimes \overrightarrow{does} \otimes \overrightarrow{not} \otimes \overrightarrow{like} \otimes \overrightarrow{Bob} —$$



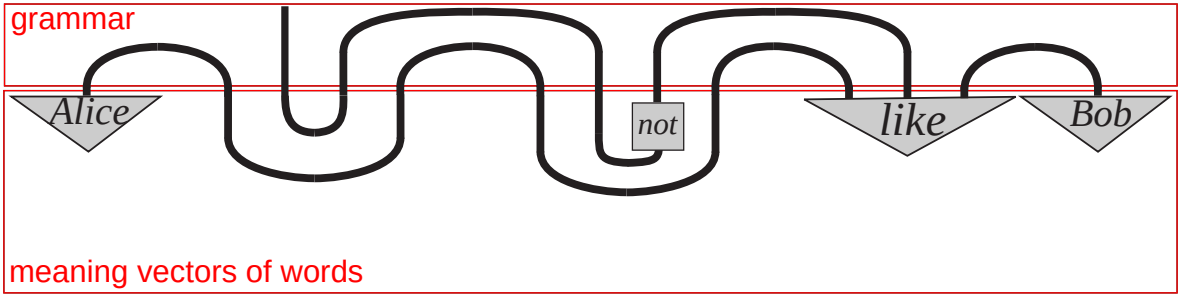
$$- \overrightarrow{Alice} \otimes \overrightarrow{does} \otimes \overrightarrow{not} \otimes \overrightarrow{like} \otimes \overrightarrow{Bob} -$$

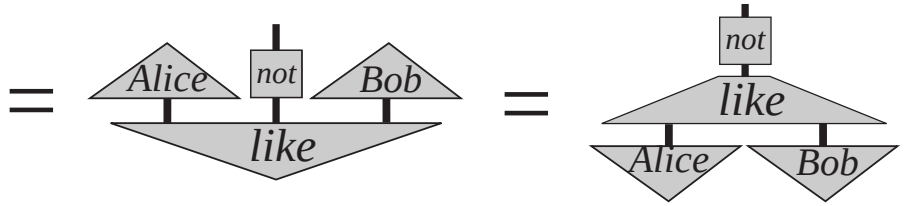


$$- \overrightarrow{Alice} \otimes \overrightarrow{does} \otimes \overrightarrow{not} \otimes \overrightarrow{like} \otimes \overrightarrow{Bob} -$$



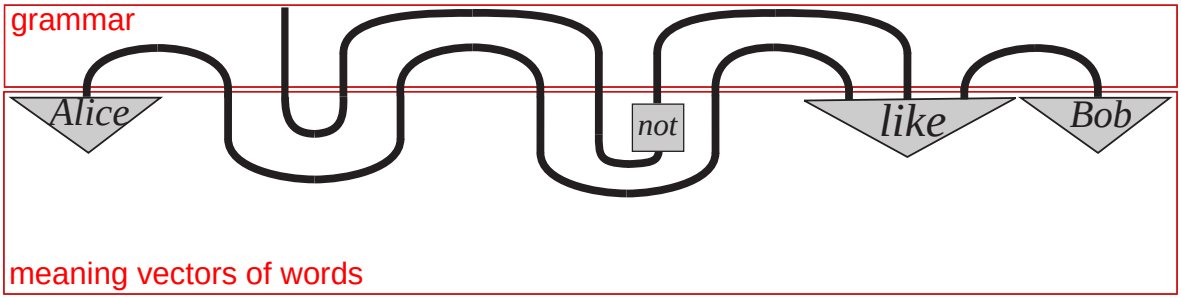
$$\overrightarrow{Alice} \otimes \overrightarrow{does} \otimes \overrightarrow{not} \otimes \overrightarrow{like} \otimes \overrightarrow{Bob}$$

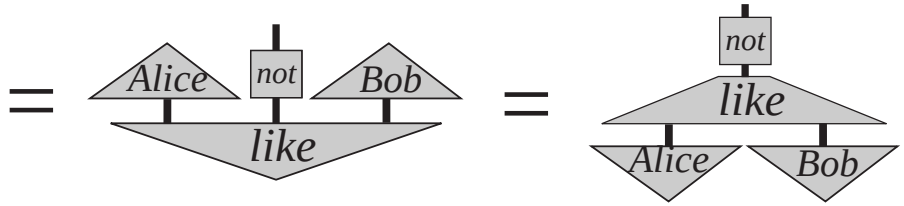




Using:

$$\overrightarrow{Alice} \otimes \overrightarrow{does} \otimes \overrightarrow{not} \otimes \overrightarrow{like} \otimes \overrightarrow{Bob}$$





$$= \vec{g}\left(\vec{f}(\vec{x}, \vec{y})\right)$$

— *experiment: word disambiguation* —

E.g. what is "saw"' in: "Alice saw Bob with a saw".

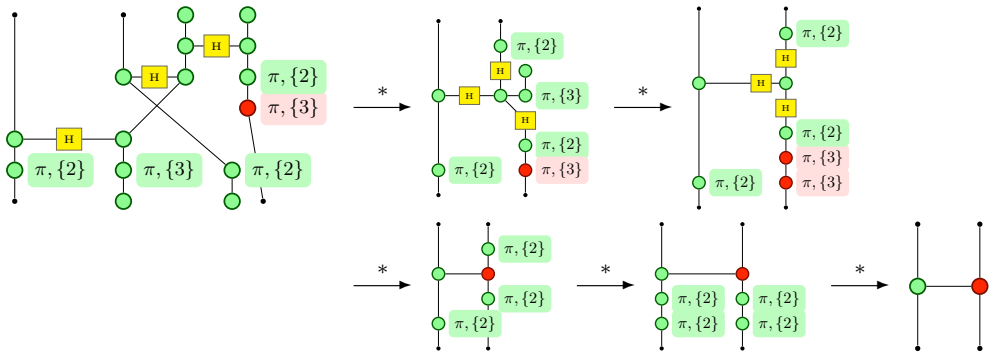
Model	High	Low	ρ
Baseline	0.47	0.44	0.16
Add	0.90	0.90	0.05
Multiply	0.67	0.59	0.17
Categorical (1)	**0.73**	**0.72**	**0.21**
Categorical (2)	**0.34**	**0.26**	**0.28**
UpperBound	4.80	2.49	0.62

Edward Grefenstette & Mehrnoosh Sadrzadeh (2011) *Experimental support for a categorical compositional distributional model of meaning*. Accepted for: Empirical Methods in Natural Language Processing (EMNLP'11).

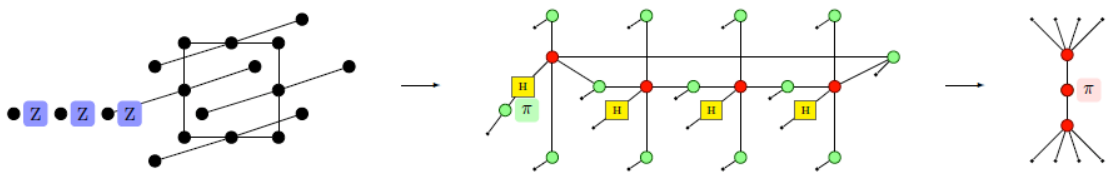
UNIVERSAL QUANTUM REASONING

— *example outputs* —

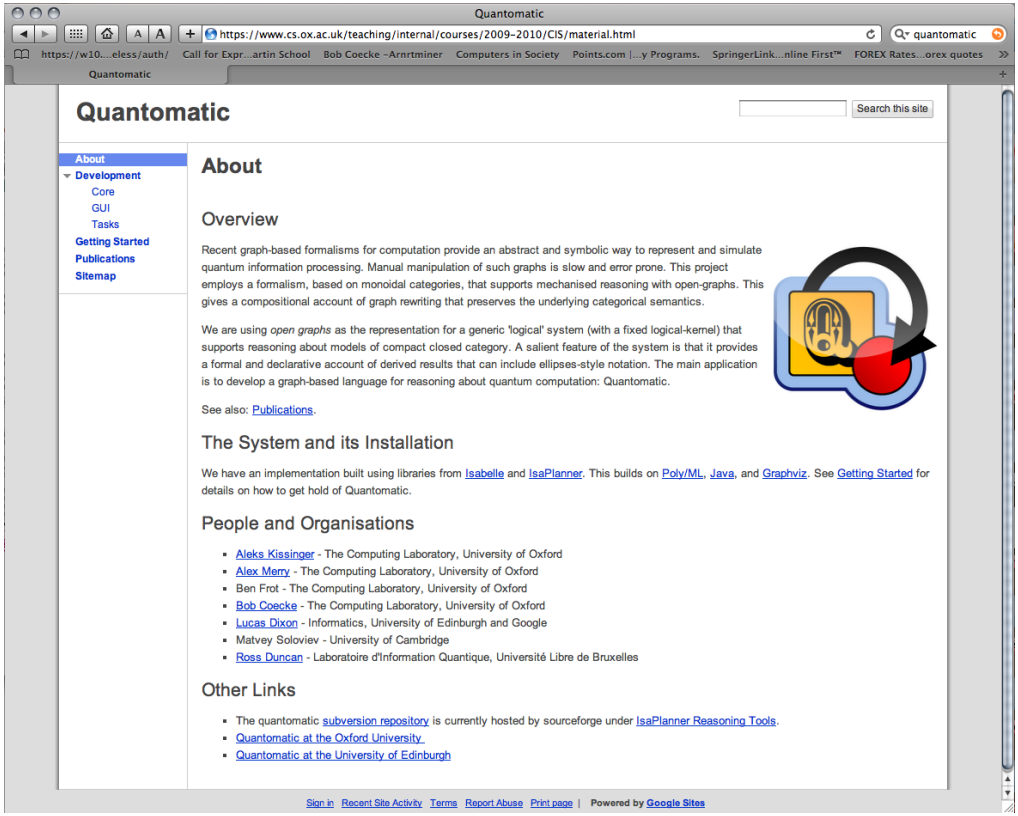
New results on resource requirements, complexity of translations in MBQC (Duncan-Perdrix ICALP'10):



Similar stuff for TMBQC (Clare Horsman NJP'11):



— *automated theory exploration* —

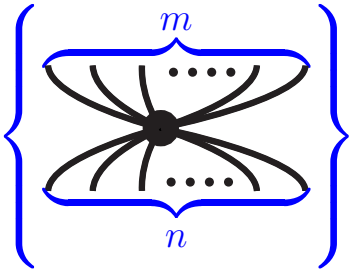


Lucas Dixon (Google), Ross Duncan (Brussels), Aleks Kissinger, Alex Merry (Oxf) and Matvey Soloviev (Camb) — `http://sites.google.com/site/quantomatic/`

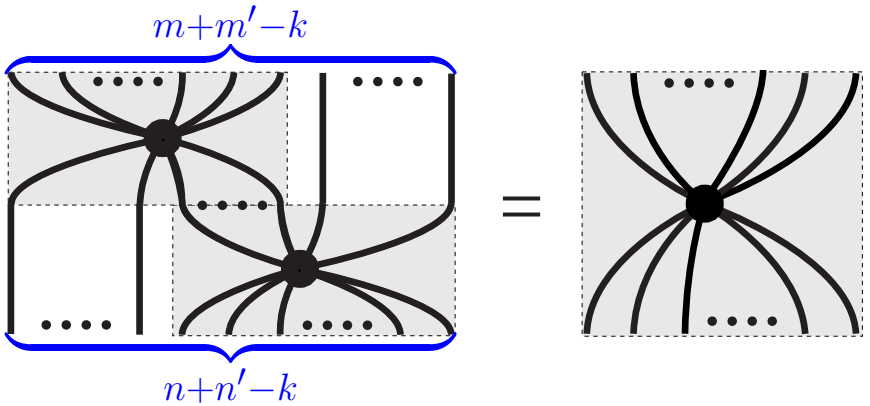
WIRES?

SPIDERS!

— *spiders* —

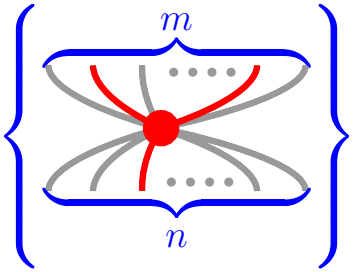
'spiders' $=$ 

such that, for $k > 0$:

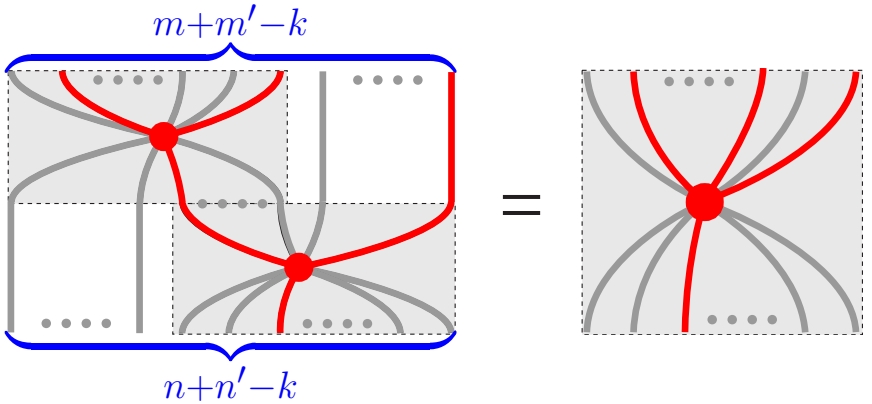


— *spiders* —

'(co-)mult.' $=$

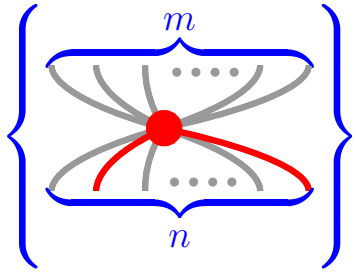


such that, for $k > 0$:

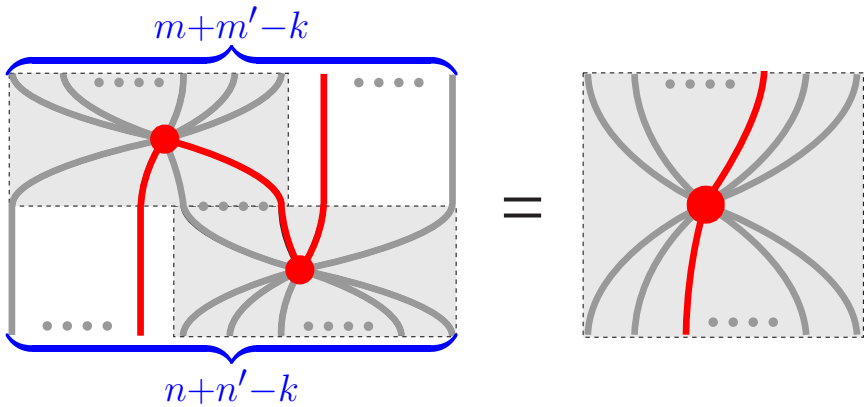


— *spiders* —

'cups/caps' $=$



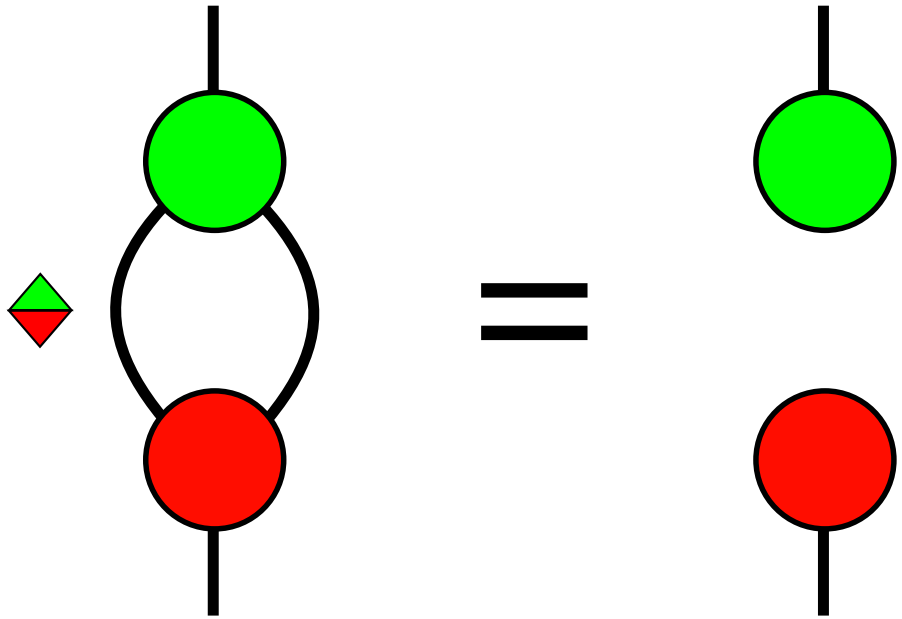
such that, for $k > 0$:



— *complementary spiders* —

— *complementary spiders* —

Thm.



BC & Ross Duncan (2008) *Interacting quantum observables*. ICALP'08 & New Journal of Physics **13**, 043016. `arXiv:0906.4725`