Experiments on growth series of braid groups

Jean Fromentin

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Words

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- Definitions :
  - the empty word is denoted by ε ;
  - a word on the alphabet S is an S-word ;
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  - for $u \in S^*$: $|u|$ its length and $\overline{u}$ the element of M it represents ;
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  - $\overline{aba} = (1\ 2) \circ (2\ 3) \circ (1\ 2) = (1\ 2) \circ (1\ 3\ 2) = (1\ 3)$,

    $\overline{bab} = (2\ 3) \circ (1\ 2) \circ (2\ 3) = (2\ 3) \circ (1\ 2\ 3) = (1\ 3)$

    and so $aba \equiv bab$.

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The spherical growth series of M w.r.t. S is

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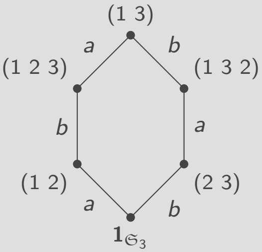

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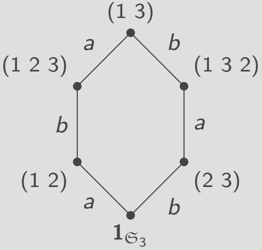

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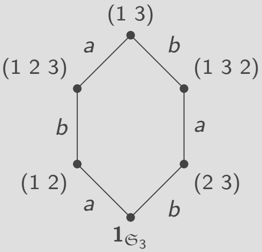

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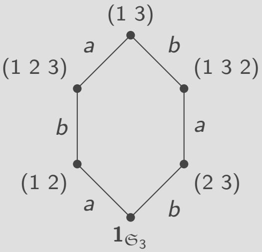

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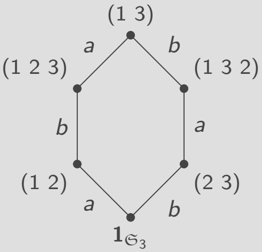

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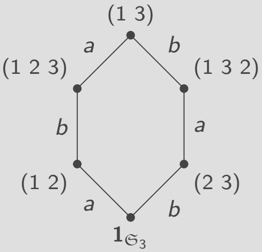

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$$s(\mathfrak{S}_3, S, \ell) = \begin{cases} 1 & \text{if } \ell = 0 \text{ or } \ell = 3, \\ 2 & \text{if } \ell = 1 \text{ or } \ell = 2, \\ 0 & \text{if } \ell \geqslant 4. \end{cases}$$

and so $\mathcal{S}(\mathfrak{S}_3, S) = 1 + 2t + 2t^2 + t^3$.

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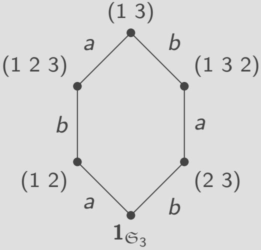

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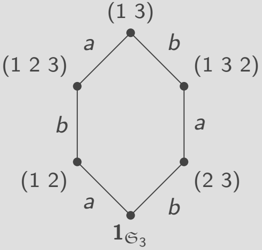

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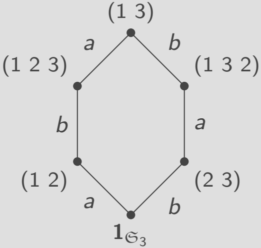

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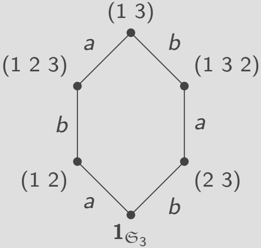

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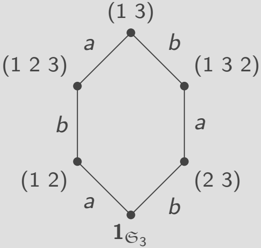

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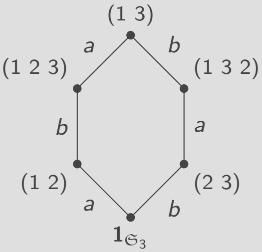

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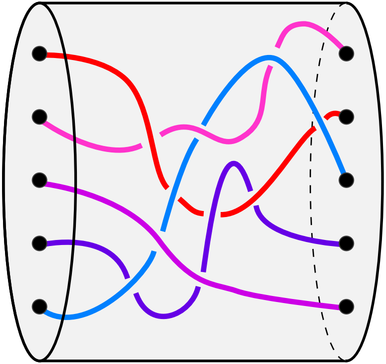

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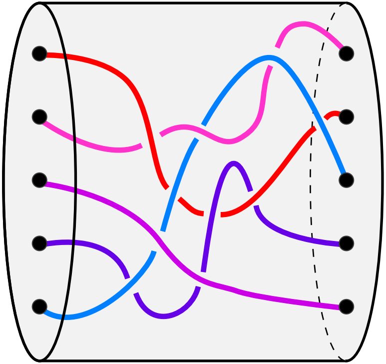

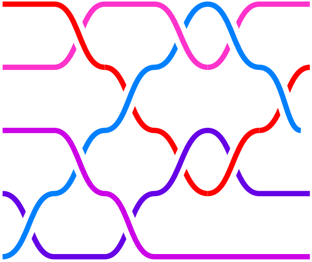

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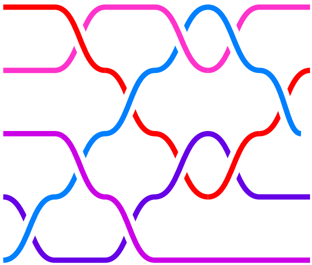

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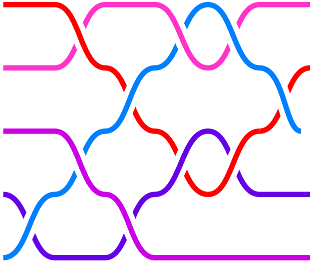

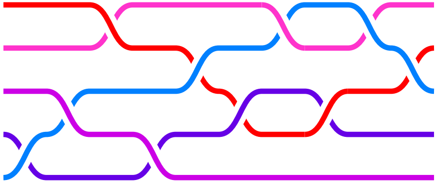

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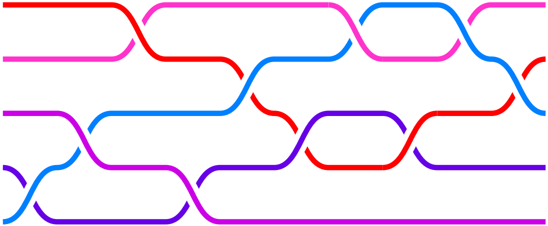

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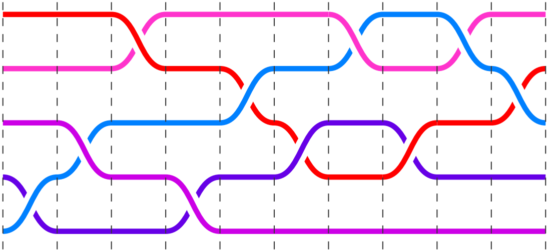

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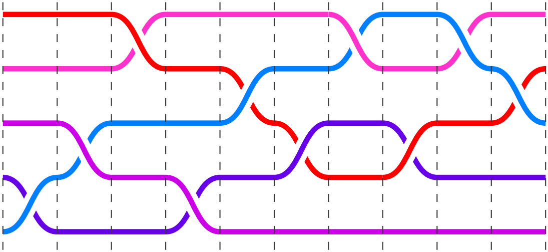
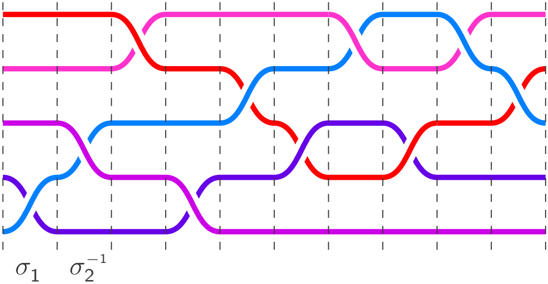


σ_1

Braid words



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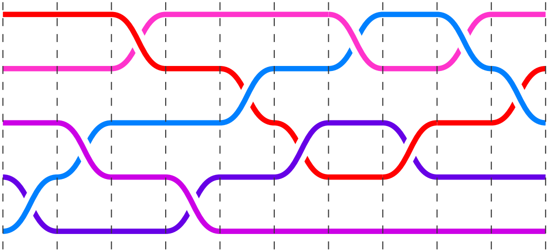

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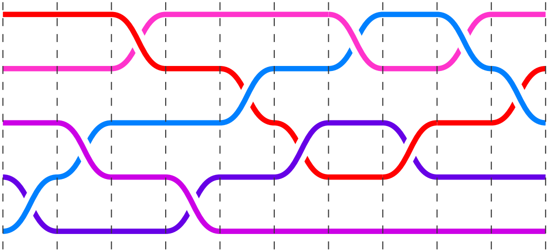

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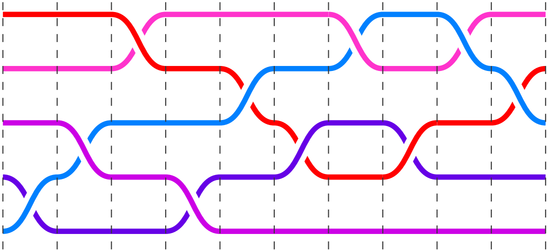

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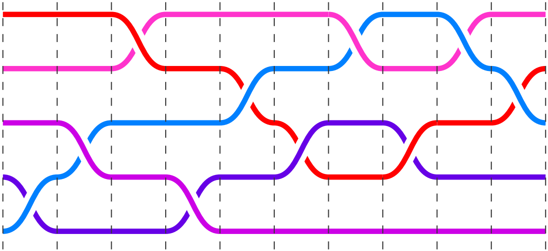

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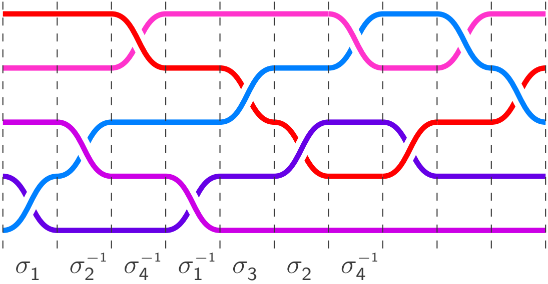

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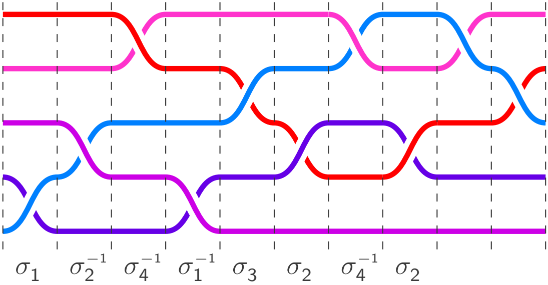
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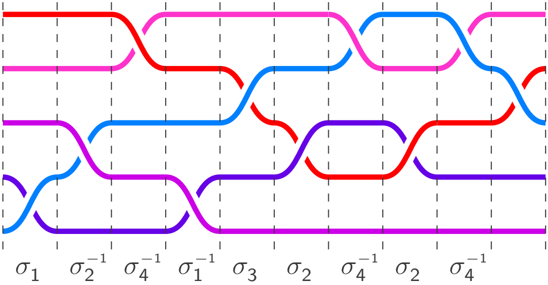
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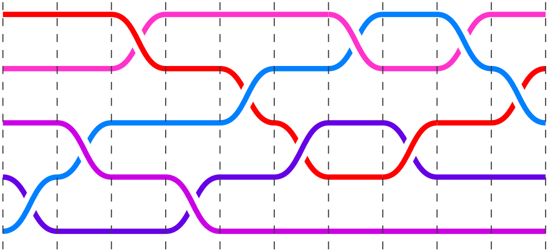

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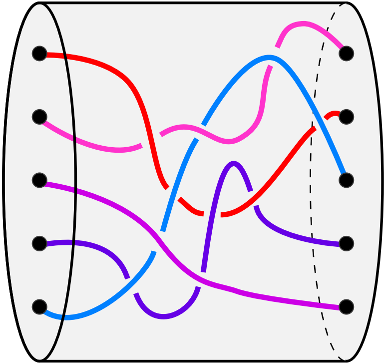

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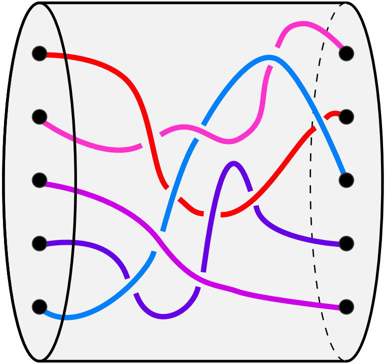

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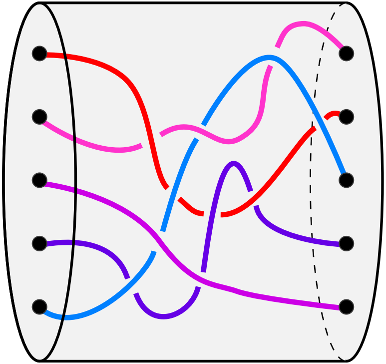

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Artin's presentation of braid groups

- Theorem (E. Artin 1925) : The braid group B_n is presented by

$$\left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{rcll} \sigma_i\sigma_j & \equiv & \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i & \equiv & \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle \tag{1}$$

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- Open question : Do $\mathcal{G}(B_n, \Sigma_n)$ and $\mathcal{S}(B_n, \Sigma_n)$ are rationals for $n \geqslant 4$?

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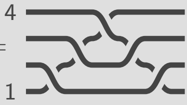
$a_{1,4} =$  $\approx$

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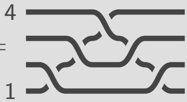
$a_{1,4} =$  $\approx$  $\leftrightarrow$

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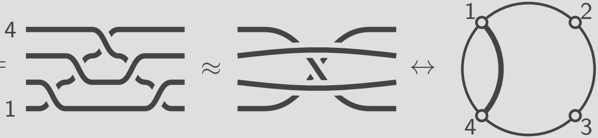
$a_{1,4} =$ 

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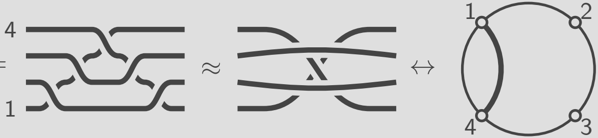
$a_{1,4} =$ 

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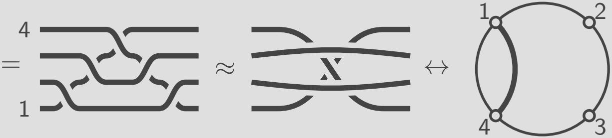

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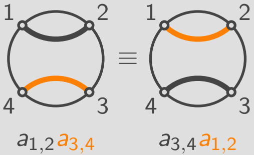

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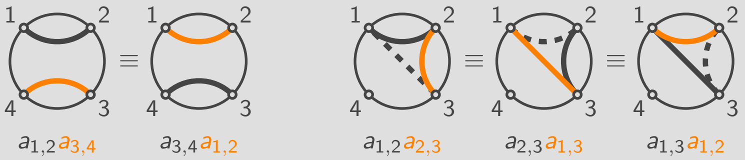

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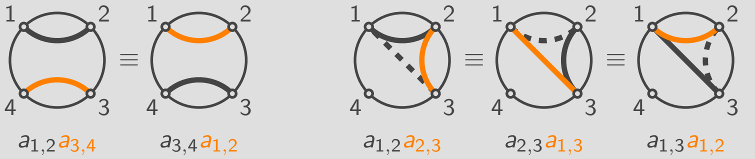

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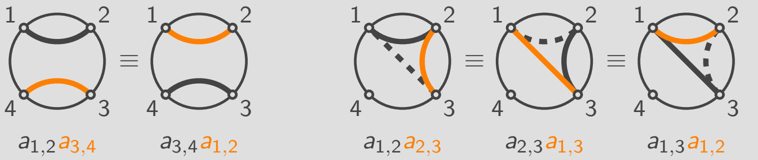

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  - $s(B_n, S_n; \ell) = \mathrm{card}\left(\{\beta \in B_n \text{ s.t. } |\beta|_{S_n} = \ell\}\right)$,
  - $g(B_n, S_n; \ell) = \mathrm{card}\left(\{u \in S_n^* \text{ s.t. } |u| = |\overline{u}|_{S_n} = \ell\}\right)$.

  - Efficient as possible, in particular in the case of $n \leqslant 4$.
  - Here we focus on $s(B_n, S_n; \ell)$.

- Compute as many terms as possible of $s(B_n, S_n; \ell)$ and $g(B_n, S_n; \ell)$ with $n = 4$ for $S_n = \Sigma_n$ and $n = 3, 4$ for $S_n = \Sigma_n^*$.

- Try to guess rational values for
  - $\mathcal{S}(B_4, \Sigma_4)$ and $\mathcal{G}(B_4, \Sigma_4)$,
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  - No good algorithms.

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Assume $\ell \geqslant 2$.

- Lemma : Let u be a geodesic S_n-word of length $\ell - 1$ and $\alpha \in S_n$.
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  - W_ℓ from W_ℓ' by keeping only one word in each $\equiv$-classes.

A first algorithm

- Algorithm $(W_{\ell-2} \vdash B_n(S_n; \ell-2),\ W_{\ell-1} \vdash B_n(S_n; \ell-1))$:

for $u \in W_{\ell-1}$ **do**
    for $\alpha \in S_n$ **do**
        $v \leftarrow u\,\alpha$
        if $v \not\trianglelefteq W_{\ell-2}$ and $v \not\trianglelefteq W_\ell$ **then**
            $W_\ell \leftarrow W_\ell \sqcup \{v\}$ *A new braid* $\overline{v}$ *of* $B_n(S_n; \ell)$ *is found.*
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- Question : How to test if a S_n-word u appears in a subset W of S_n^* ?

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  - Better but we can do more in this direction.

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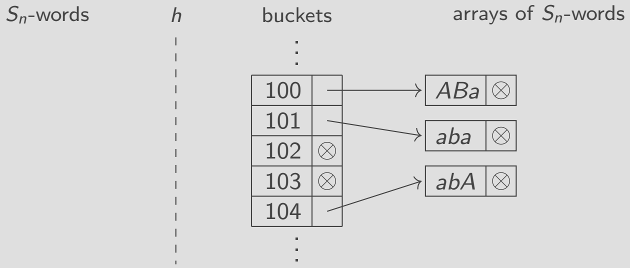

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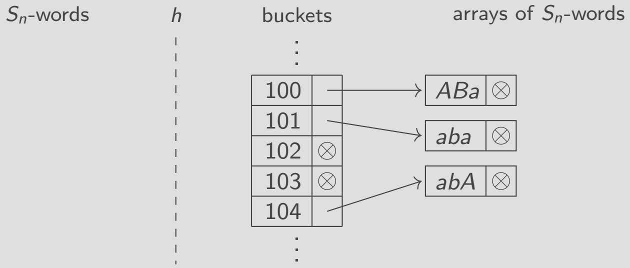

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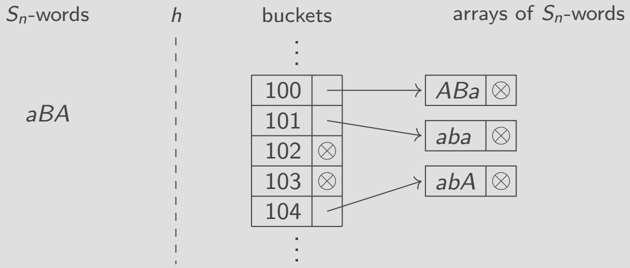

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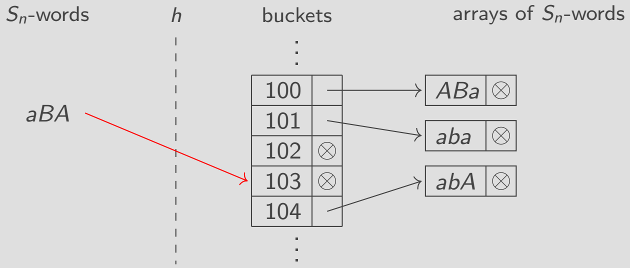

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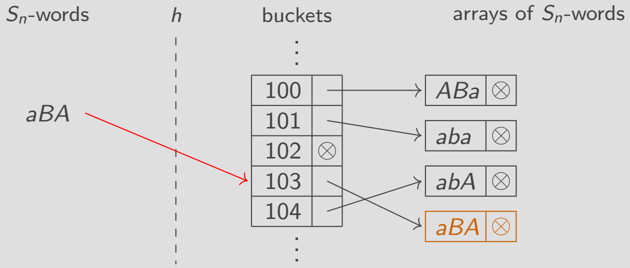

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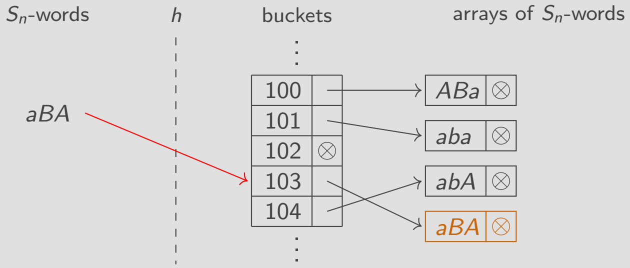

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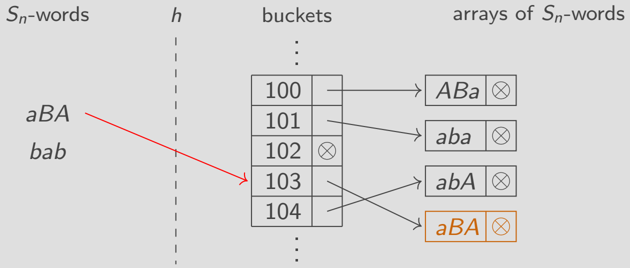

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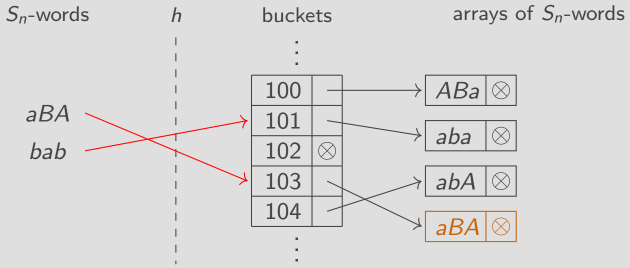

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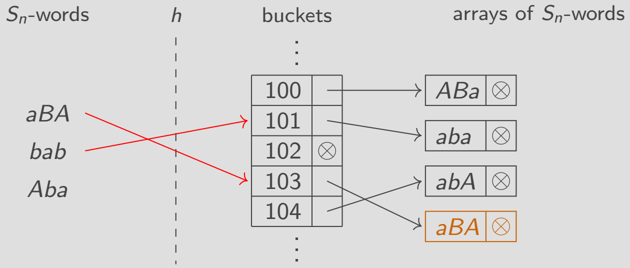

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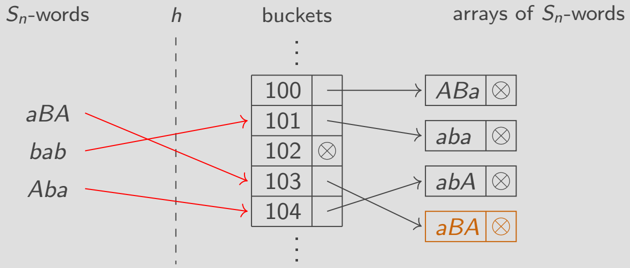

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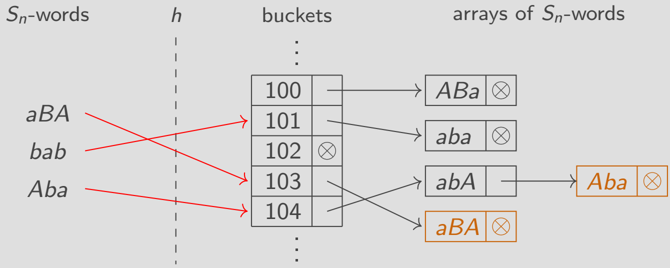

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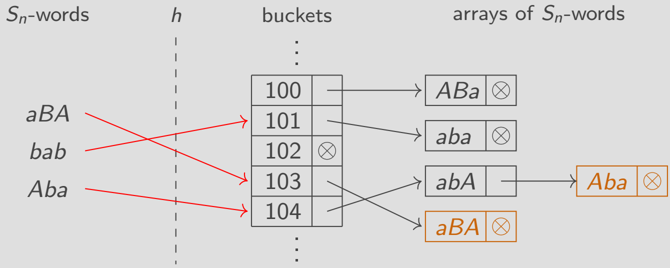

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- In average case, insertion has a constant time complexity and a linear one in worst case.

Dynnikov's coordinates

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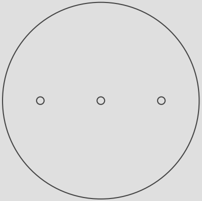

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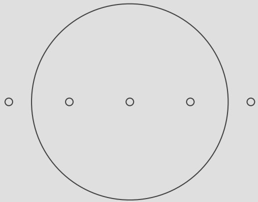

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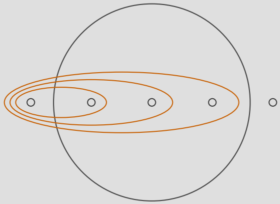

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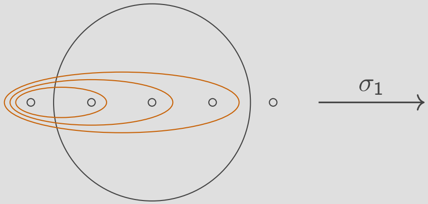

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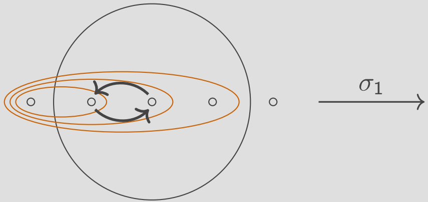

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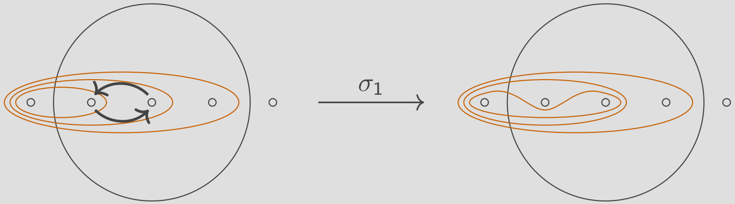

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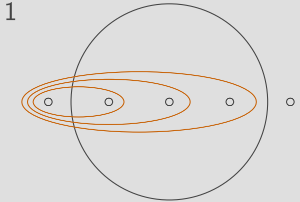

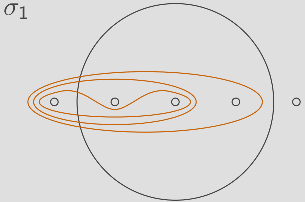

Dynnikov's coordinates

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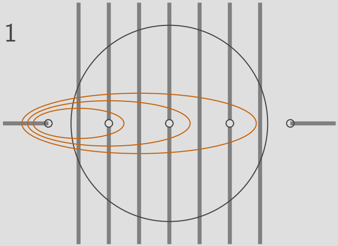
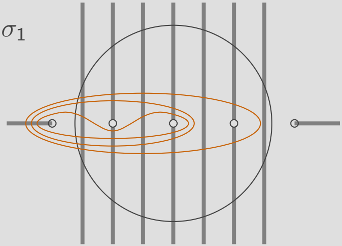


σ_1



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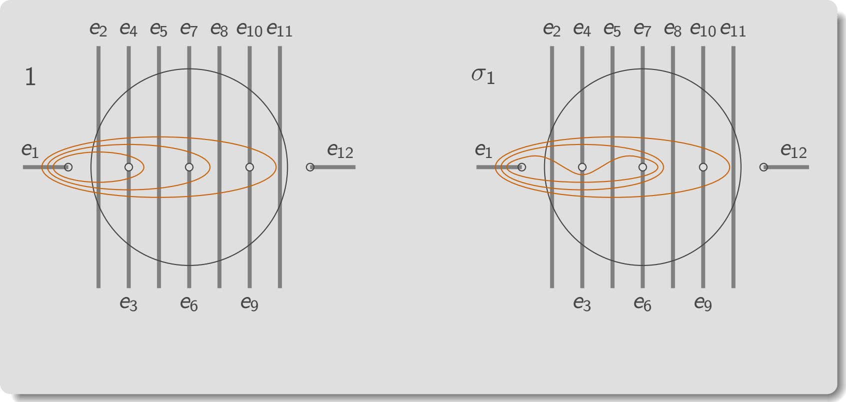

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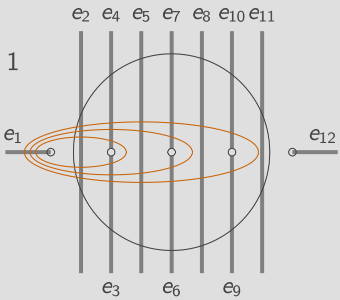

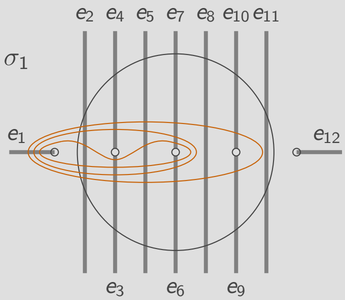

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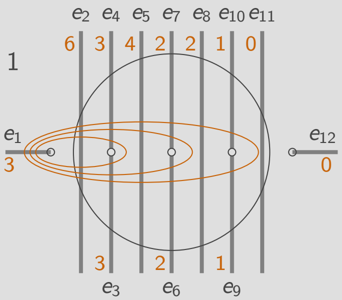

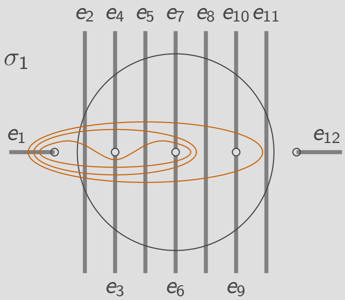

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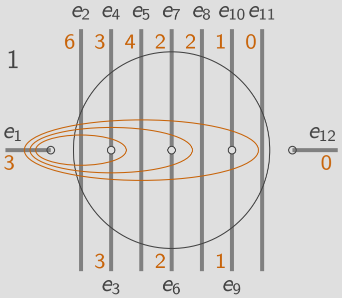

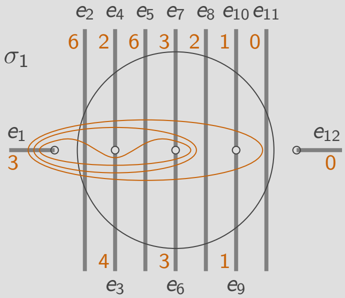

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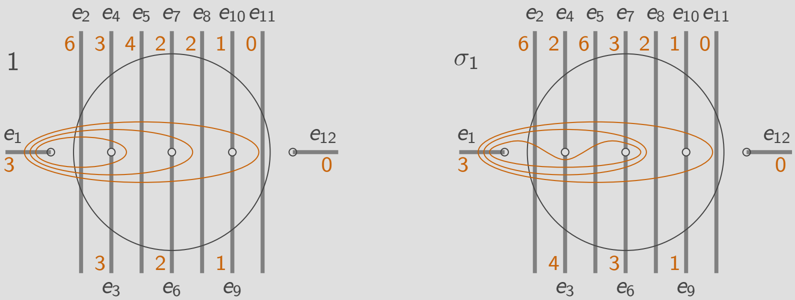

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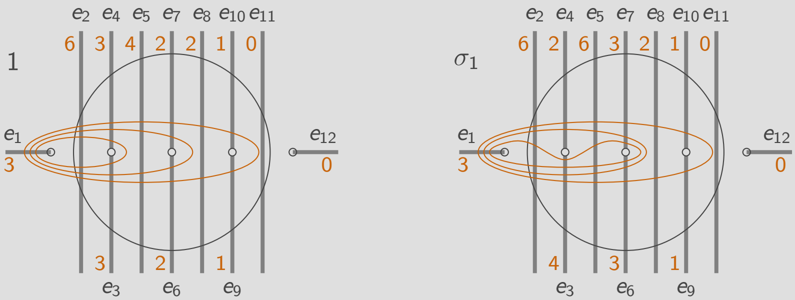

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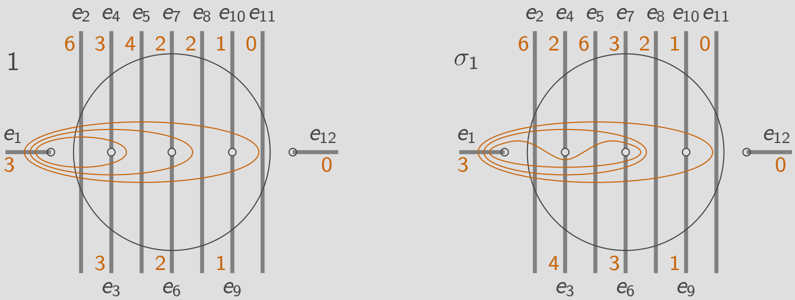

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- **Examples** : $\rho_D(1) = (0, 1, 0, 1, 0, 1)$ and $\rho_D(\sigma_1) = (1, 0, 0, 2, 0, 1)$.

Dynnikov's coordinates – Properties

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$$h(u) = \sum_{i=0}^{7} \operatorname{rem}(c_i, 256)\ 256^i$$
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    - well-suited for 64-bits computers.

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  - Allow parallelization.

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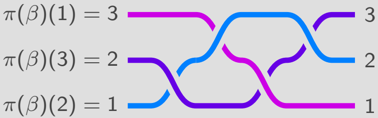

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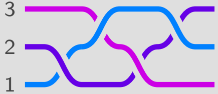

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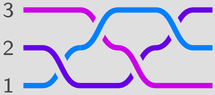

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- **Corollary** : For $\alpha \in S_n$, $\ell_{i,j}(\beta \cdot \alpha)$ depends only of $\ell_{*,*}(\beta)$ and $\pi(\beta)$.

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- Definition : The template of $\beta \in B_n$ is

$$\tau(\beta) = (\pi(\beta), \ell_{1,2}(\beta), \ldots, \ell_{n-1,n}(\beta)) \in \mathfrak{S}_n \times \mathbb{Z}^{\frac{n(n+1)}{2}}$$

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- **Fact** :
$$B_n(S_n;\ell,t) = \bigcup_{\alpha \in S_n} \{\beta \cdot \alpha \text{ with } \beta \in B_n(S_n;\ell-1, t \odot \alpha^{-1})\}$$

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- Algorithm (Storing a representative set of $B_n(S_n, \ell, t)$) :

$W_{\ell,t} \leftarrow \emptyset$
$W_{\ell-2,t} \leftarrow \mathtt{Load}(\ell-2, t)$
for $\alpha \in S_n$ **do**
  $t' \leftarrow t \odot \alpha^{-1}$
  $W_{\ell-1,t'} \leftarrow \mathtt{Load}(\ell-1, t')$
  for $u \in W_{\ell-1,t'}$ **do**
    $v \leftarrow u\,\alpha$
    if $v \not\triangleleft W_{\ell-2,t}$ and $v \not\triangleleft W_{\ell,t}$ **then**
      $W_{\ell,t} \leftarrow W_{\ell,t} \sqcup \{v\}$
    end if
  end for
end for
$\mathtt{Save}(W_{\ell,t}, \ell, t)$

Stable bijection

- Definition : A bijection μ of S_n^* is S_n-stable whenever :
  - μ preserves the word length,
  - for all u, v in S_n^* we have $\mu(u) \equiv \mu(v) \Leftrightarrow u \equiv v$
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- We have the following diagram

$$\begin{tikzcd}
S_n^\ell \arrow[r, "\mu"] \arrow[d, "\tau"'] & S_n^\ell \arrow[d, "\tau"] \\
T_n(S_n;\ell) \arrow[r, "\mu^T"'] & T_n(S_n;\ell)
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  ▶ It is sufficient to compute only one of these sets.

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$$\mathrm{inv}_{\Sigma_n}(x_1 \cdots x_\ell) = (x_\ell^{-1} \cdots x_1^{-1})$$

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$$\mathrm{inv}_{\Sigma_n}(x_1 \cdots x_\ell) = (x_\ell^{-1} \cdots x_1^{-1})$$

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- Lemma : The subgroups G_{Σ_n} of bijections of $T_n(\Sigma_n;\ell)$ generated by $\{\mathrm{inv}^T_{\Sigma_n}, \mathrm{mir}^T_{\Sigma_n}, \Phi^T_{\Sigma_n}\}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$.

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Template reduction

S_n denotes either Σ_n or Σ_n^*

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- Proposition :

$$s(B_n, S_n; \ell) = \sum_{t \in T_n(S_n;\ell)} \mathrm{card}\left(B_n(S_n; \ell, t)\right)$$

$$= \sum_{\substack{t \in T_n(S_n;\ell) \\ t \text{ reduced}}} \mathrm{card}\left(B_n(S_n; \ell, t)\right) \times \mathrm{card}\left(G_{S_n} \star t\right)$$

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▶ Can be efficitively used from an algorithmic point of view.

Experimentation

- Implementation : Distribued `C++` code based on a clients / server model.

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- Validation of L. Sabalka formulas :

$$\mathcal{S}(B_3,\Sigma_3) = \frac{(t+1)(2t^3-t^2+t-1)}{(t-1)(2t-1)(t^2+t-1)},$$
$$\mathcal{G}(B_3,\Sigma_3) = \frac{t^4+3t^3+t+1}{(t^2+2t-1)(t^2+t-1)}.$$

Three strand – Dual case

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ℓ	$s(B_3, \Sigma_3^*; \ell)$	$g(B_3, \Sigma_3^*; \ell)$	ℓ	$s(B_3, \Sigma_3^*; \ell)$	$g(B_3, \Sigma_3^*; \ell)$
0	1	1	11	38 910	6 639 606
1	6	6	12	83 966	26 216 418
2	20	30	13	180 222	103 827 366
3	54	126	14	385 022	412 169 970
4	134	498	15	819 198	1 639 212 246
5	318	1 926	16	1 736 702	6 528 347 778
6	734	7 410	17	3 670 014	26 027 690 886
7	1 662	28 566	18	7 733 246	103 853 269 650
8	3 710	110 658	19	16 252 926	414 639 810 486
9	8 190	431 046	20	34 078 718	1 656 237 864 738
10	17 918	1 687 890	21	71 303 166	6 617 984 181 606

- Conjecture :

$$\mathcal{S}(B_3, \Sigma_3^*) = \frac{(t+1)(2t^2-1)}{(t-1)(2t-1)^2}, \quad \mathcal{G}(B_3, \Sigma_3^*) = \frac{12t^3 - 2t^2 + 3t - 1}{(2t-1)(3t-1)(4t-1)}.$$

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  - With growth rates of 2 and 4 respectively.

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ℓ	$s(B_4, \Sigma_4; \ell)$	$g(B_4, \Sigma_4; \ell)$	ℓ	$s(B_4, \Sigma_4; \ell)$	$g(B_4, \Sigma_4; \ell)$
0	1	1	13	9 007 466	281 799 158
1	6	6	14	27 218 486	1 153 638 466
2	26	30	15	82 133 734	4 710 108 514
3	98	142	16	247 557 852	19 186 676 438
4	338	646	17	745 421 660	78 004 083 510
5	1 110	2 870	18	2 242 595 598	316 591 341 866
6	3 542	12 558	19	6 741 618 346	1 283 041 428 650
7	11 098	54 026	20	20 252 254 058	5 193 053 664 554
8	34 362	229 338	21	60 800 088 680	20 994 893 965 398
9	105 546	963 570	22	182 422 321 452	84 795 261 908 498
10	322 400	4 016 674	23	547 032 036 564	342 173 680 884 002
11	980 904	16 641 454	24	1 639 548 505 920	1 379 691 672 165 334
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- No good conjectures.

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- No good conjectures.
- The storage of all braids of B_4 with geodesic Σ_4-length $\leqslant 25$

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- No good conjectures.
- The storage of all braids of B_4 with geodesic Σ_4-length $\leqslant 25$ requires 26 To of disk space.

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0	1	1	9	7 348 366	708 368 540
1	12	12	10	35 773 324	6 128 211 364
2	84	132	11	173 885 572	52 826 999 612
3	478	1 340	12	844 277 874	454 136 092 148
4	2 500	12 788	13	4 095 929 948	3 895 624 824 092
5	12 612	117 452	14	19 858 981 932	33 359 143 410 468
6	62 570	1 053 604	15	96 242 356 958	285 259 736 104 444
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- Conjecture :

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- ▶ The growth rate of $\mathcal{S}(B_4,\Sigma_4^*)$ is $\simeq 4.8$.

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- ▶ The growth rate of $\mathcal{S}(B_4, \Sigma_4^*)$ is $\simeq 4.8$.
- ▶ No good conjecture for $\mathcal{G}(B_4, \Sigma_4^*)$.

Thank you !